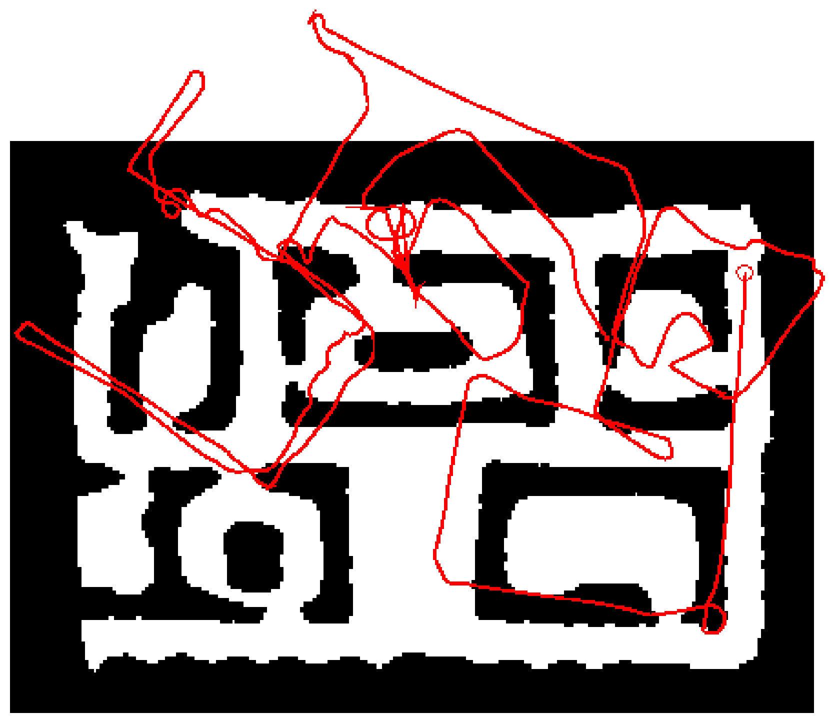
Probabilistic Motion Models

Nived Chebrolu

Slide courtesy to C. Stachniss, W. Burgard and D. Fox

Basic Motion Models

- Motion is inherently uncertain
- How can we model this uncertainty?



Courtesy: Thrun, Burgard, Fox

Motion Model

$$p(x_t \mid u_t, x_{t-1})$$

State Estimation

- Estimate the state x of a system given observations z and controls u
- **Goal:**

$$p(x \mid z, u)$$

Recursive Bayes Filter

$$bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$

$$= \quad \eta\, p(z_t \mid x_t) \int \underbrace{p(x_t \mid x_{t-1}, u_t)}_{\textbf{motion model}}\, bel(x_{t-1})\, dx_{t-1}$$

Motion and Observation Model

- Prediction step

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid u_t, x_{t-1})}_{} \, bel(x_{t-1}) \, dx_{t-1}$$

motion model

- Correction step

$$bel(x_t) = \eta \, \underbrace{p(z_t \mid x_t)}_{} \, \overline{bel}(x_t)$$

observation model
(also: measurement or sensor model)

Probabilistic Motion Models

- Specifies a posterior probability that action *u* carries the robot from x_{t-1} to x_t

$$p(x_t \mid u_t, x_{t-1})$$

Typical Motion Models

- Two common types of motion models:
  - **Odometry-based**
  - **Velocity-based (dead reckoning)**

- **Odometry-based models** are used when systems are equipped with wheel encoders
- They calculate the new pose based on the encoder values
- **Velocity-based models** have to be applied when no wheel encoders are given
- They calculate the new pose based on the velocities and the time elapsed

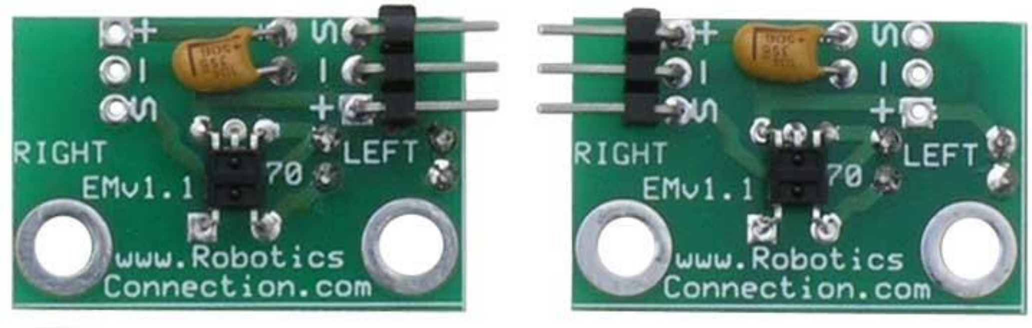
Dead Reckoning

- Derived from "deduced reckoning"
- Procedure for determining the current location of a vehicle
- Achieved by calculating the current pose of the vehicle based on its velocities and the time elapsed
- Historically used to log the position of ships

[Image source: Wikipedia, LoKiLeCh]

Example Wheel Encoders

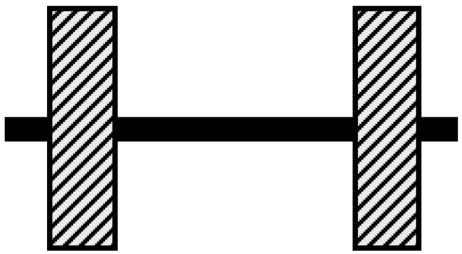
These modules provide +5V output when they "see" white, and a 0V output when they "see" black.



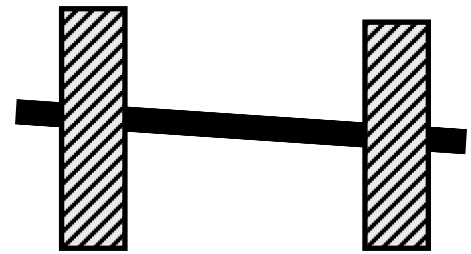
These disks are manufactured out of high quality laminated color plastic to offer a very crisp black to white transition. This enables a wheel encoder sensor to easily see the transitions.

Source: http://www.active-robots.com/

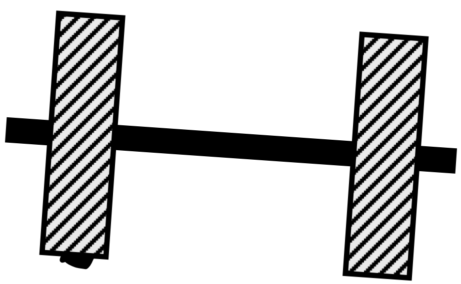
Reasons for Motion Errors of Wheeled Robots



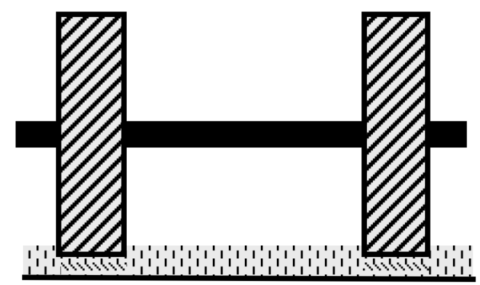
ideal case



different wheel diameters



bump



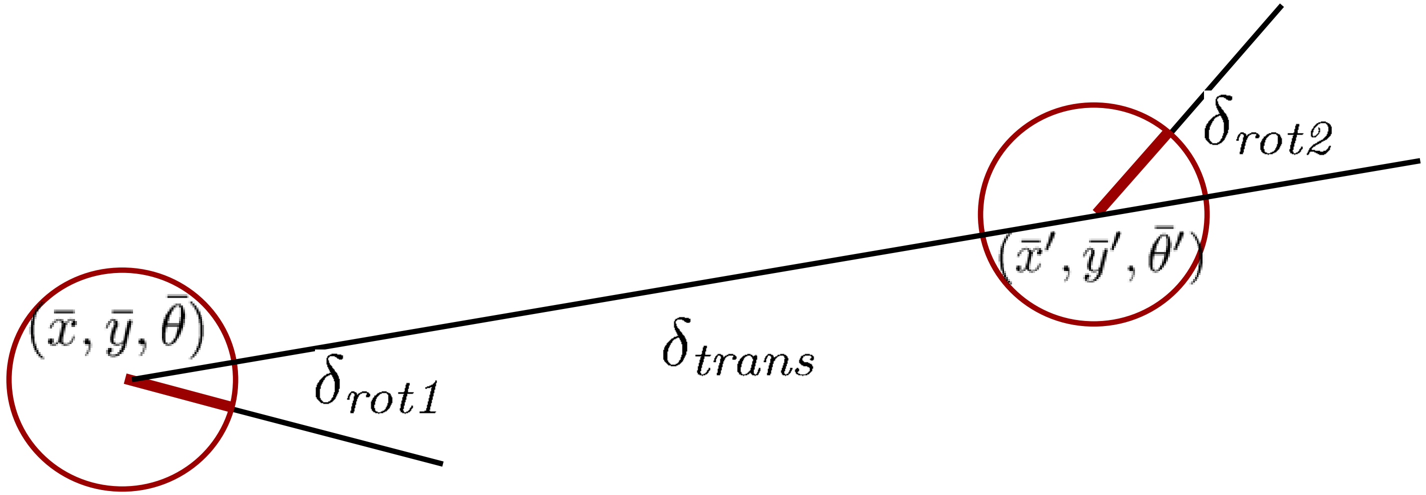
carpet

and many more …

Odometry Model

- Motion from $(\bar{x}, \bar{y}, \bar{\theta})$ to $(\bar{x}', \bar{y}', \bar{\theta}')$
- Odometry information $u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$

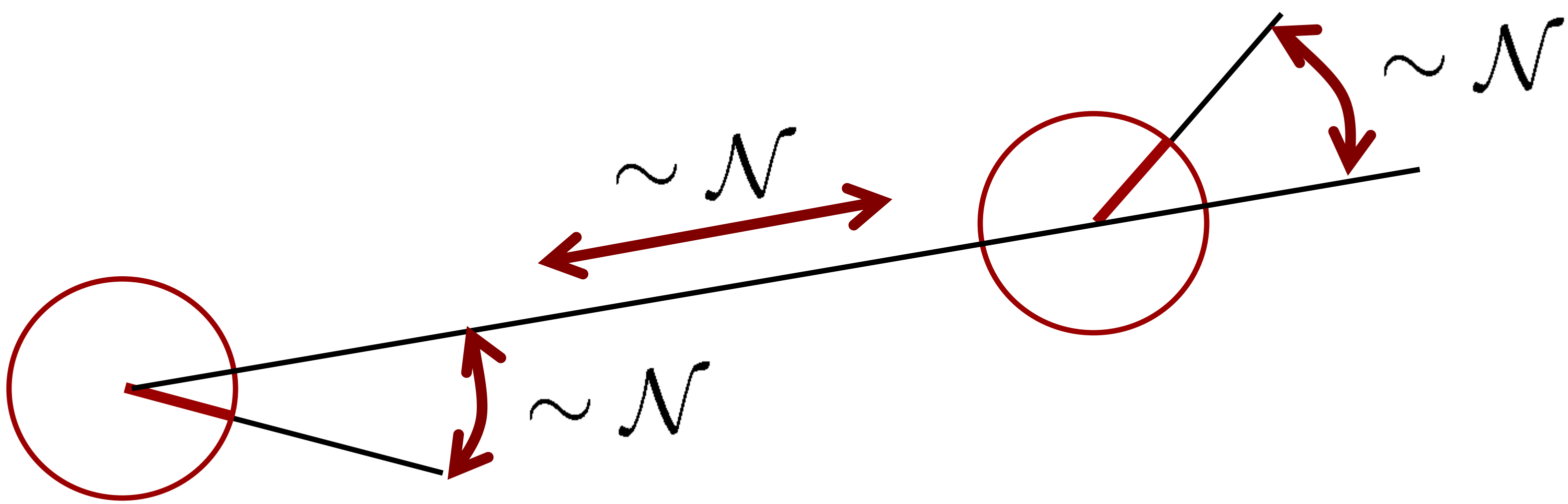
$$
\begin{aligned}
\delta_{trans} &= \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2} \\
\delta_{rot1} &= \operatorname{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta} \\
\delta_{rot2} &= \bar{\theta}' - \bar{\theta} - \delta_{rot1}
\end{aligned}
$$



Probability Distribution

- Noise in odometry $u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$
- Example: Gaussian noise

$$u \sim \mathcal{N}(0, \Sigma)$$



Noise Model for Odometry

- The measured motion is given by the true motion corrupted with noise.

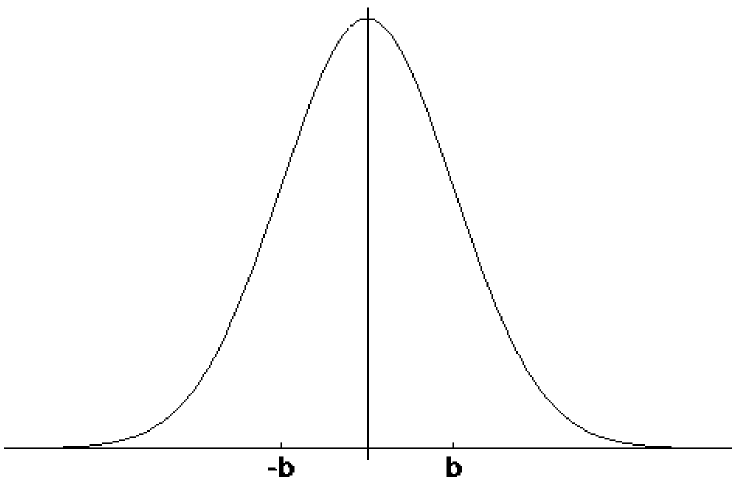
$$\hat{\delta}_{rot1} = \delta_{rot1} + \varepsilon_{\alpha_1 |\delta_{rot1}| + \alpha_2 |\delta_{trans}|}$$

$$\hat{\delta}_{trans} = \delta_{trans} + \varepsilon_{\alpha_3 |\delta_{trans}| + \alpha_4 (|\delta_{rot1}| + |\delta_{rot2}|)}$$

$$\hat{\delta}_{rot2} = \delta_{rot2} + \varepsilon_{\alpha_1 |\delta_{rot2}| + \alpha_2 |\delta_{trans}|}$$

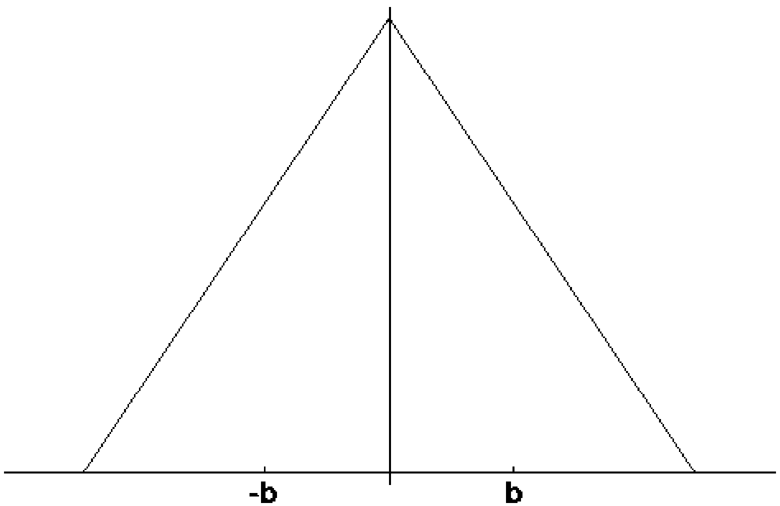
Typical Distributions for Probabilistic Motion Models

Normal distribution



$$\varepsilon_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

Triangular distribution



$$\varepsilon_{\sigma^2}(x) = \begin{cases} 0 \text{ if } |\mathrm{x}| > \sqrt{6\sigma^2} \\ \dfrac{\sqrt{6\sigma^2} - |x|}{6\sigma^2} \end{cases}$$

Calculating the Probability Density (zero-centered)

- For a normal distribution

query point

1. **Algorithm prob_normal_distribution**(a,b):

std. deviation

2. **return** $\frac{1}{\sqrt{2\pi\ b^2}}\ \exp\left\{-\frac{1}{2}\frac{a^2}{b^2}\right\}$

- For a triangular distribution

1. **Algorithm prob_triangular_distribution**(a,b):
2. **return** $\max\left\{0, \frac{1}{\sqrt{6}\ b} - \frac{|a|}{6\ b^2}\right\}$

Calculating the Posterior Given x, x’, and Odometry

hypotheses odometry

1. **Algorithm motion_model_odometry** $(\boxed{x, x'}, \boxed{\bar{x}, \bar{x}'})$
2. $\delta_{trans} = \sqrt{(\bar{x}'-\bar{x})^2 + (\bar{y}'-\bar{y})^2}$
3. $\delta_{rot1} = \text{atan2}(\bar{y}'-\bar{y}, \bar{x}'-\bar{x}) - \bar{\theta}$
4. $\delta_{rot2} = \bar{\theta}'-\bar{\theta} - \delta_{rot1}$

odometry params (u)

5. $\hat{\delta}_{trans} = \sqrt{(x'-x)^2 + (y'-y)^2}$
6. $\hat{\delta}_{rot1} = \text{atan2}(y'-y, x'-x) - \theta$
7. $\hat{\delta}_{rot2} = \theta'-\theta - \hat{\delta}_{rot1}$

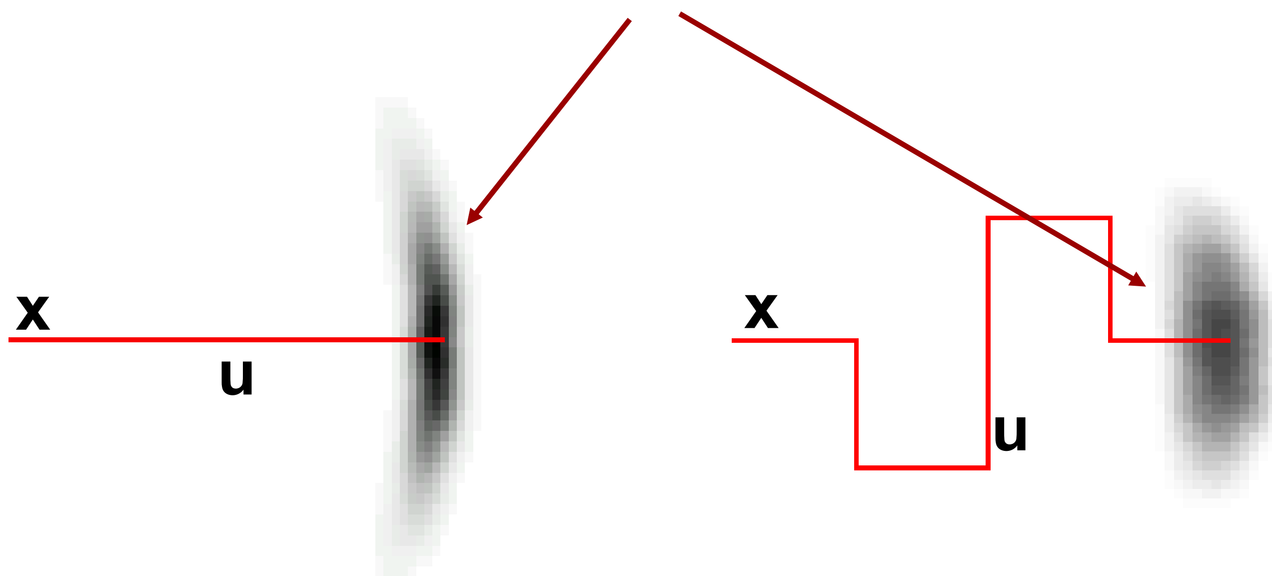
values of interest (x,x’)

8. $p_1 = \text{prob}(\delta_{\text{rot1}} - \hat{\delta}_{\text{rot1}}, \alpha_1 |\delta_{\text{rot1}}| + \alpha_2 \delta_{\text{trans}})$
9. $p_2 = \text{prob}(\delta_{\text{trans}} - \hat{\delta}_{\text{trans}}, \alpha_3 \delta_{\text{trans}} + \alpha_4 (|\delta_{\text{rot1}}| + |\delta_{\text{rot2}}|))$
10. $p_3 = \text{prob}(\delta_{\text{rot2}} - \hat{\delta}_{\text{rot2}}, \alpha_1 |\delta_{\text{rot2}}| + \alpha_2 \delta_{\text{trans}})$
11. **return** $p_1 \cdot p_2 \cdot p_3$

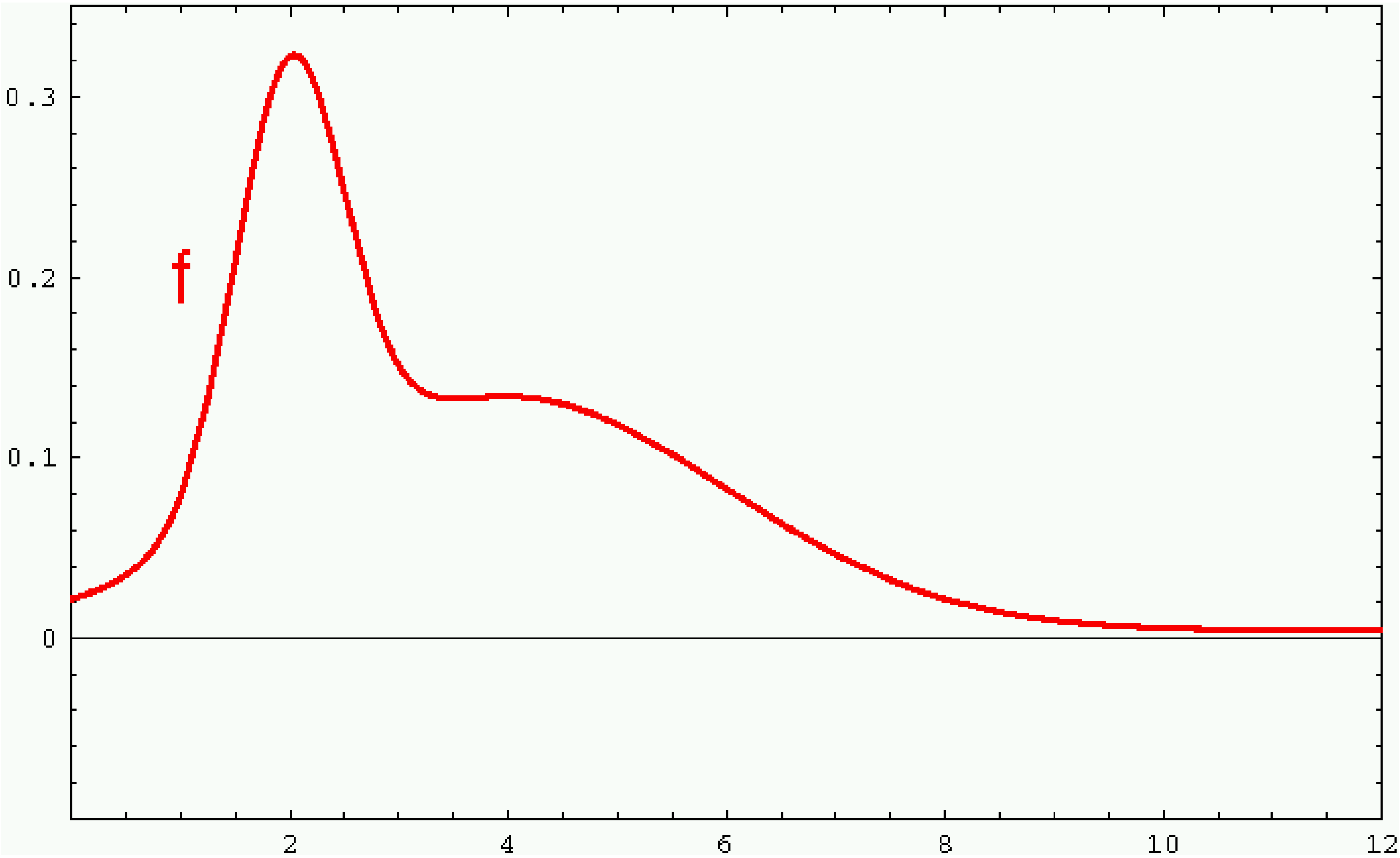
Application

- Repeated application of the motion model for short movements.
- Typical banana-shaped distributions obtained for the 2d-projection of the 3d posterior.

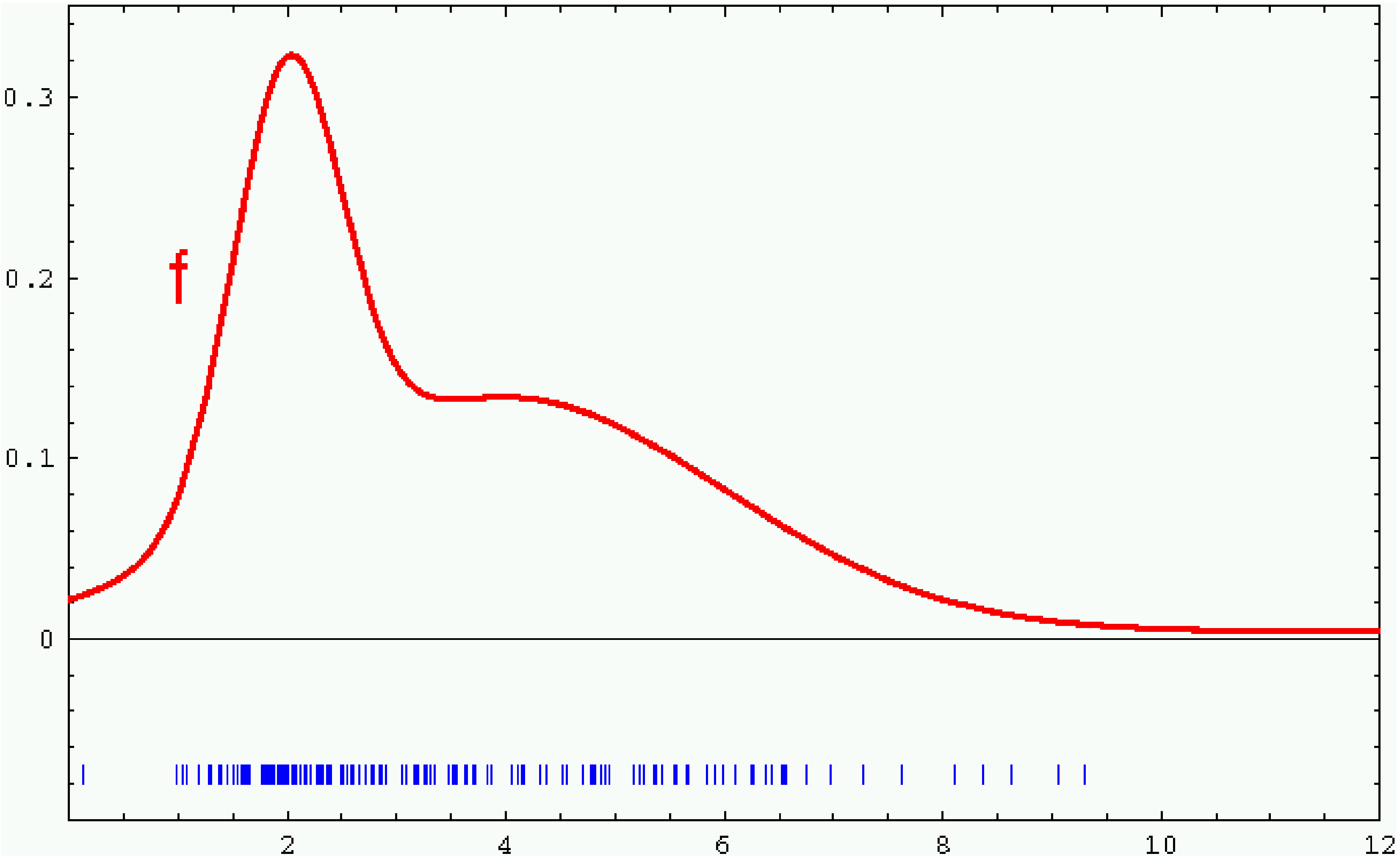
$$p(\boldsymbol{x}' \mid \boldsymbol{u}, \boldsymbol{x})$$



Sample-Based Density Representation



Sample-Based Density Representation

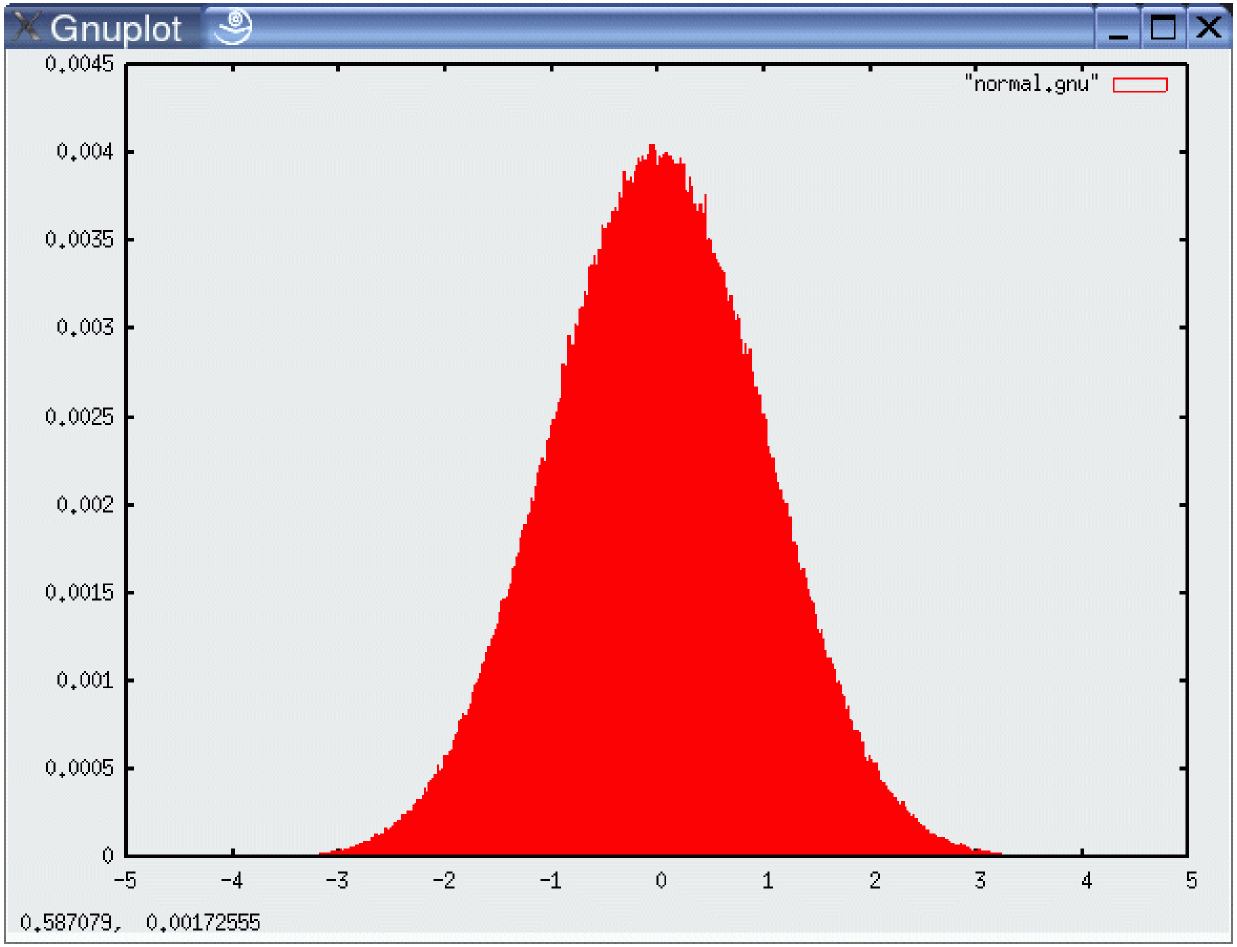


How to Sample from a Normal Distribution?

- Sampling from a normal distribution

1. **Algorithm sample_normal_distribution(b):**
2. **return** $\frac{1}{2}\sum_{i=1}^{12} rand(-b, b)$

Normally Distributed Samples



10^6 samples

How to Sample from Normal or Triangular Distributions?

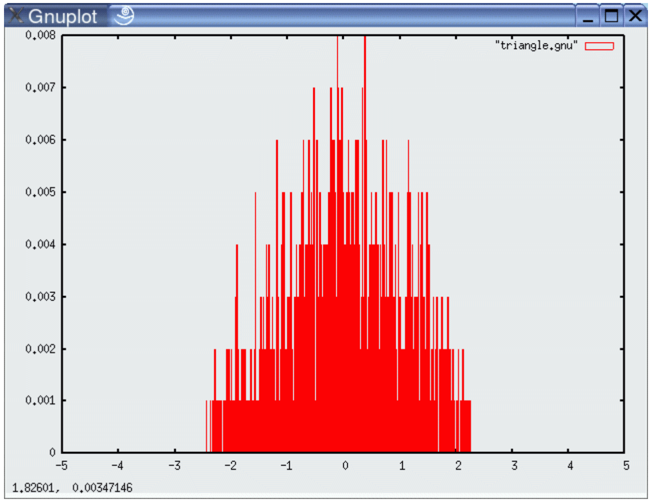
- Sampling from a normal distribution

  1. **Algorithm sample_normal_distribution(*b*):**
  2. **return** $\frac{1}{2}\sum_{i=1}^{12} rand(-b, b)$

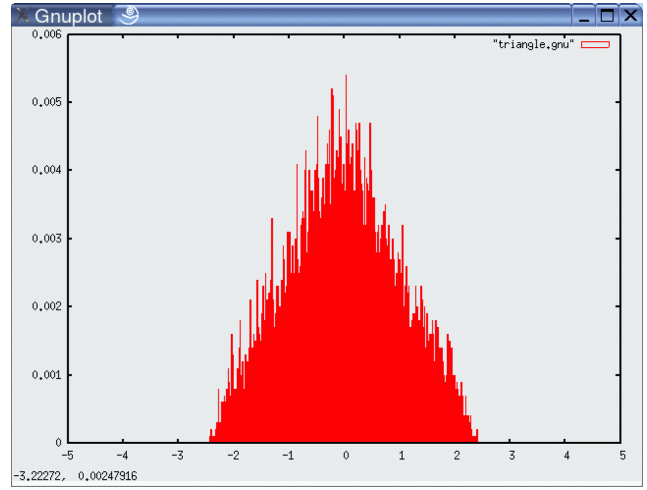
- Sampling from a triangular distribution

  1. **Algorithm sample_triangular_distribution(*b*):**
  2. **return** $\frac{\sqrt{6}}{2}\left[\mathbf{rand}(-b, b) + \mathbf{rand}(-b, b)\right]$

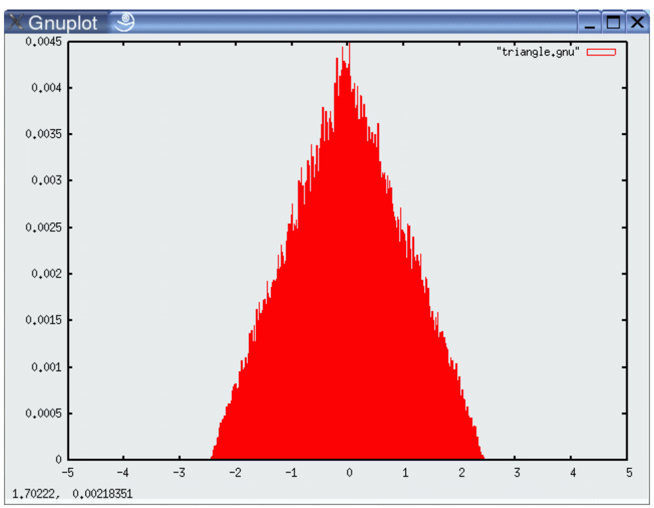
For Triangular Distribution



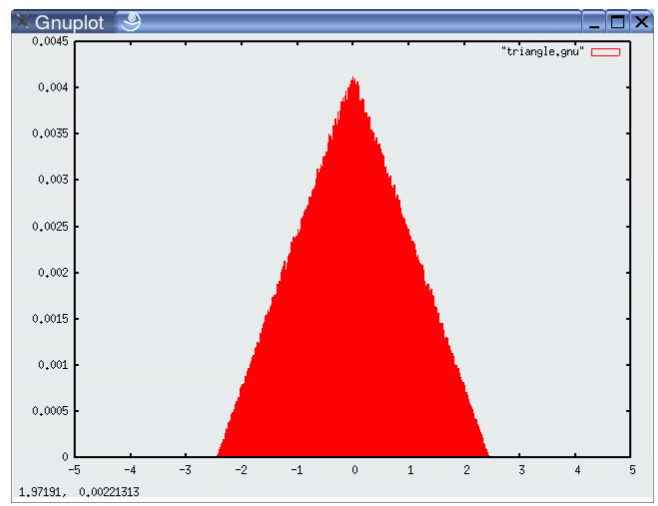
10^3 samples



10^4 samples



10^5 samples



10^6 samples

Sample Odometry Motion Model

1. **Algorithm sample_motion_model(u, x):**

   $u = \langle \delta_{rot1}, \delta_{rot2}, \delta_{trans} \rangle, x = \langle x, y, \theta \rangle$

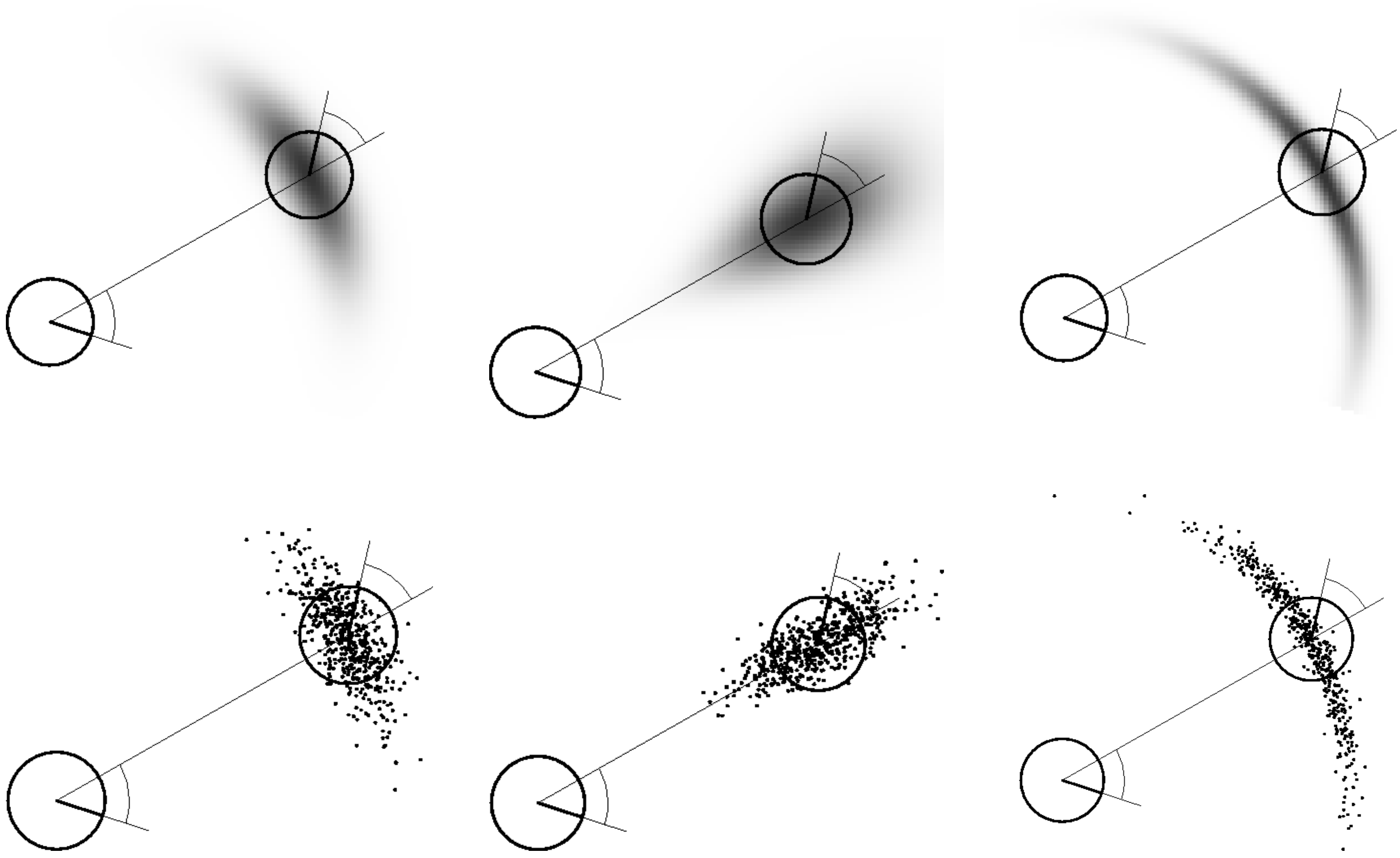
1. $\hat{\delta}_{rot1} = \delta_{rot1} + \text{sample}(\alpha_1 \mid \delta_{rot1} \mid + \alpha_2\ \delta_{trans})$
2. $\hat{\delta}_{trans} = \delta_{trans} + \text{sample}(\alpha_3\ \delta_{trans} + \alpha_4\ (\mid \delta_{rot1} \mid + \mid \delta_{rot2} \mid))$
3. $\hat{\delta}_{rot2} = \delta_{rot2} + \text{sample}(\alpha_1 \mid \delta_{rot2} \mid + \alpha_2\ \delta_{trans})$

4. $x' = x + \hat{\delta}_{trans} \cos(\theta + \hat{\delta}_{rot1})$
5. $y' = y + \hat{\delta}_{trans} \sin(\theta + \hat{\delta}_{rot1})$
6. $\theta' = \theta + \hat{\delta}_{rot1} + \hat{\delta}_{rot2}$

7. **Return** $\langle x', y', \theta' \rangle$

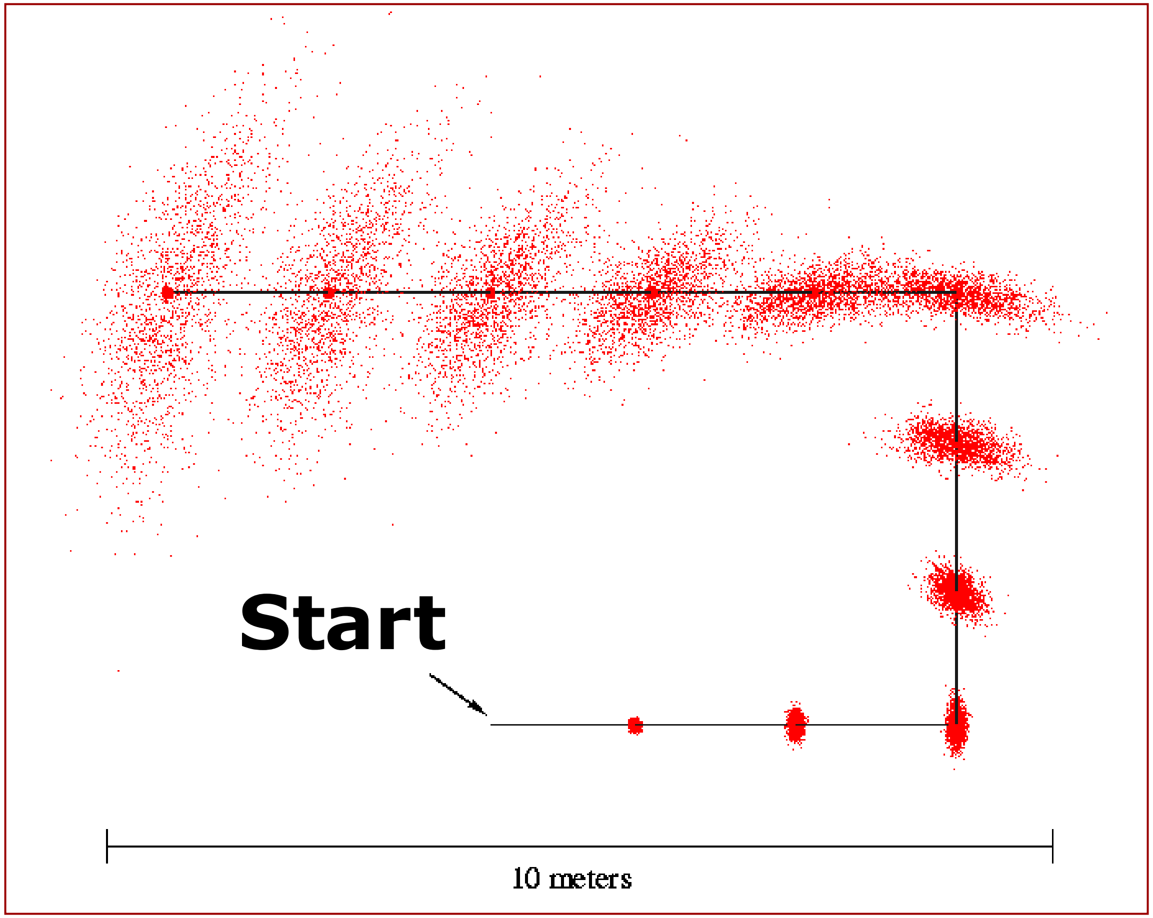
sample_normal_distribution

Examples (Odometry-Based)

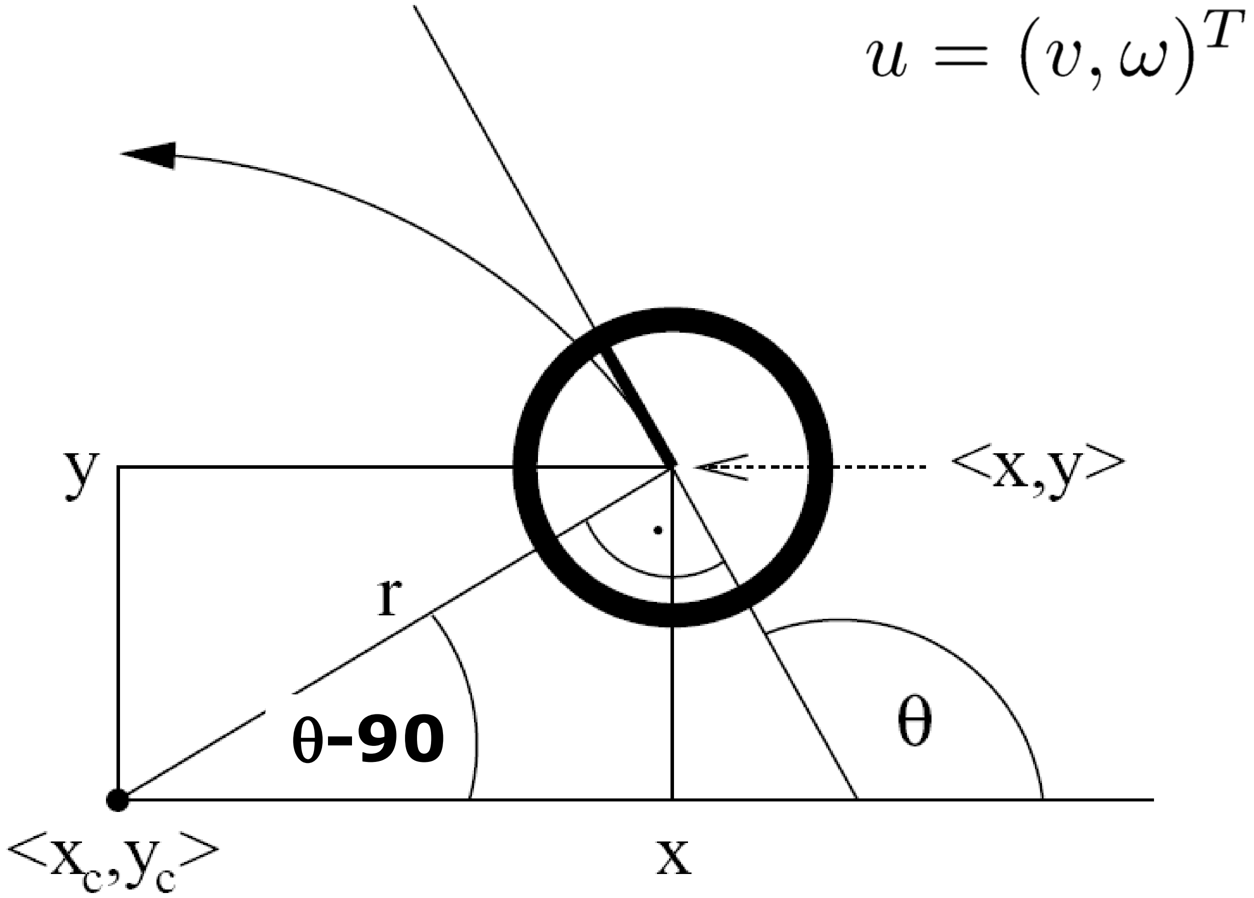


Courtesy: Thrun, Burgard, Fox

Sampling from Our Motion Model



Velocity-Based Model



Noise Model for the Velocity-Based Model

- The measured motion is given by the true motion corrupted with noise.

$$\hat{v} = v + \varepsilon_{\alpha_1|v|+\alpha_2|\omega|}$$

$$\hat{\omega} = \omega + \varepsilon_{\alpha_3|v|+\alpha_4|\omega|}$$

- Discussion: What is the disadvantage of this noise model?

Noise Model for the Velocity-Based Model

- The $(\hat{v}, \hat{\omega})$ -circle constrains the final orientation (2D manifold in 3D)
- Better approach:

$$\hat{v} = v + \varepsilon_{\alpha_1 |v| + \alpha_2 |\omega|}$$

$$\hat{\omega} = \omega + \varepsilon_{\alpha_3 |v| + \alpha_4 |\omega|}$$

$$\hat{\gamma} = \varepsilon_{\alpha_5 |v| + \alpha_6 |\omega|}$$

Term to account for the final rotation

Motion Including 3rd Parameter

$$x' = x - \frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}} \; sin(\theta + \hat{\omega}\Delta t)$$

$$y' = y + \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}} \; cos(\theta + \hat{\omega}\Delta t)$$

$$\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$$

Term to account for the final rotation

Equation for the Velocity Model

$$x_{t-1} = (x, y, \theta)^T$$
$$x_t = (x', y', \theta')^T$$

Center of circle:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin\theta \\ \lambda \cos\theta \end{pmatrix}$$

some constant (distance to ICC)

(center of circle is orthogonal to the initial heading)

See also: “Probabilistic Robotics, Chap. 5, pg. 130

Equation for the Velocity Model

$$x_{t-1} = (x, y, \theta)^T$$
$$x_t = (x', y', \theta')^T$$

some constant

Center of circle:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin\theta \\ \lambda \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y - y') \\ \frac{y+y'}{2} + \mu(x' - x) \end{pmatrix}$$

some constant (the center of the circle lies on a ray half way between x and x' and is orthogonal to the line between x and x')

See also: "Probabilistic Robotics, Chap. 5, pg. 130

Equation for the Velocity Model

$$x_{t-1} = (x, y, \theta)^T$$
$$x_t = (x', y', \theta')^T$$

some constant

Center of circle:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin\theta \\ \lambda \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y-y') \\ \frac{y+y'}{2} + \mu(x'-x) \end{pmatrix}$$

Allows us to solve the equations to:

$$\mu = \frac{1}{2} \frac{(x-x')\cos\theta + (y-y')\sin\theta}{(y-y')\cos\theta - (x-x')\sin\theta}$$

See also: “Probabilistic Robotics, Chap. 5, pg. 130

Equation for the Velocity Model

$$x_{t-1} = (x, y, \theta)^T$$
$$x_t = (x', y', \theta')^T$$

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y - y') \\ \frac{y+y'}{2} + \mu(x' - x) \end{pmatrix} \qquad \mu = \frac{1}{2} \frac{(x - x')\cos\theta + (y - y')\sin\theta}{(y - y')\cos\theta - (x - x')\sin\theta}$$

and

$$r^* = \sqrt{(x^* - x)^2 + (y^* - y)^2}$$
$$\Delta\theta = \operatorname{atan2}(y' - y^*, x' - x^*) - \operatorname{atan2}(y - y^*, x - x^*)$$

See also: “Probabilistic Robotics, Chap. 5, pg. 130

Equation for the Velocity Model

- The parameters of the circle:

$$r^* = \sqrt{(x^* - x)^2 + (y^* - y)^2}$$
$$\Delta\theta = \mathrm{atan2}(y' - y^*, x' - x^*) - \mathrm{atan2}(y - y^*, x - x^*)$$

- allow for computing the velocities as

$$\begin{aligned} v &= \frac{\Delta\theta}{\Delta t} r^* \\ \omega &= \frac{\Delta\theta}{\Delta t} \end{aligned}$$

See also: “Probabilistic Robotics, Chap. 5, pg. 130

Posterior Probability for Velocity Model

Algorithm motion_model_velocity(x_t, u_t, x_{t-1})**:** $p(x_t \mid x_{t-1}, u_t)$

$$\mu = \frac{1}{2} \frac{(x - x')\cos\theta + (y - y')\sin\theta}{(y - y')\cos\theta - (x - x')\sin\theta}$$

$$x^* = \frac{x + x'}{2} + \mu(y - y')$$

$$y^* = \frac{y + y'}{2} + \mu(x' - x)$$

$$r^* = \sqrt{(x - x^*)^2 + (y - y^*)^2}$$

$$\Delta\theta = \text{atan2}(y' - y^*, x' - x^*) - \text{atan2}(y - y^*, x - x^*)$$

$$\hat{v} = \frac{\Delta\theta}{\Delta t}\, r^*$$

$$\hat{\omega} = \frac{\Delta\theta}{\Delta t}$$

$$\hat{\gamma} = \frac{\theta' - \theta}{\Delta t} - \hat{\omega}$$

return $\mathbf{prob}(v - \hat{v}, \alpha_1 v^2 + \alpha_2 \omega^2) \cdot \mathbf{prob}(\omega - \hat{\omega}, \alpha_3 v^2 + \alpha_4 \omega^2) \cdot \mathbf{prob}(\hat{\gamma}, \alpha_5 v^2 + \alpha_6 \omega^2)$

Sampling from Velocity Model

Algorithm sample_motion_model_velocity(u_t, x_{t-1})**:**

$\hat{v} = v + \textbf{sample}(\alpha_1 v^2 + \alpha_2 \omega^2)$

$\hat{\omega} = \omega + \textbf{sample}(\alpha_3 v^2 + \alpha_4 \omega^2)$

$\hat{\gamma} = \textbf{sample}(\alpha_5 v^2 + \alpha_6 \omega^2)$

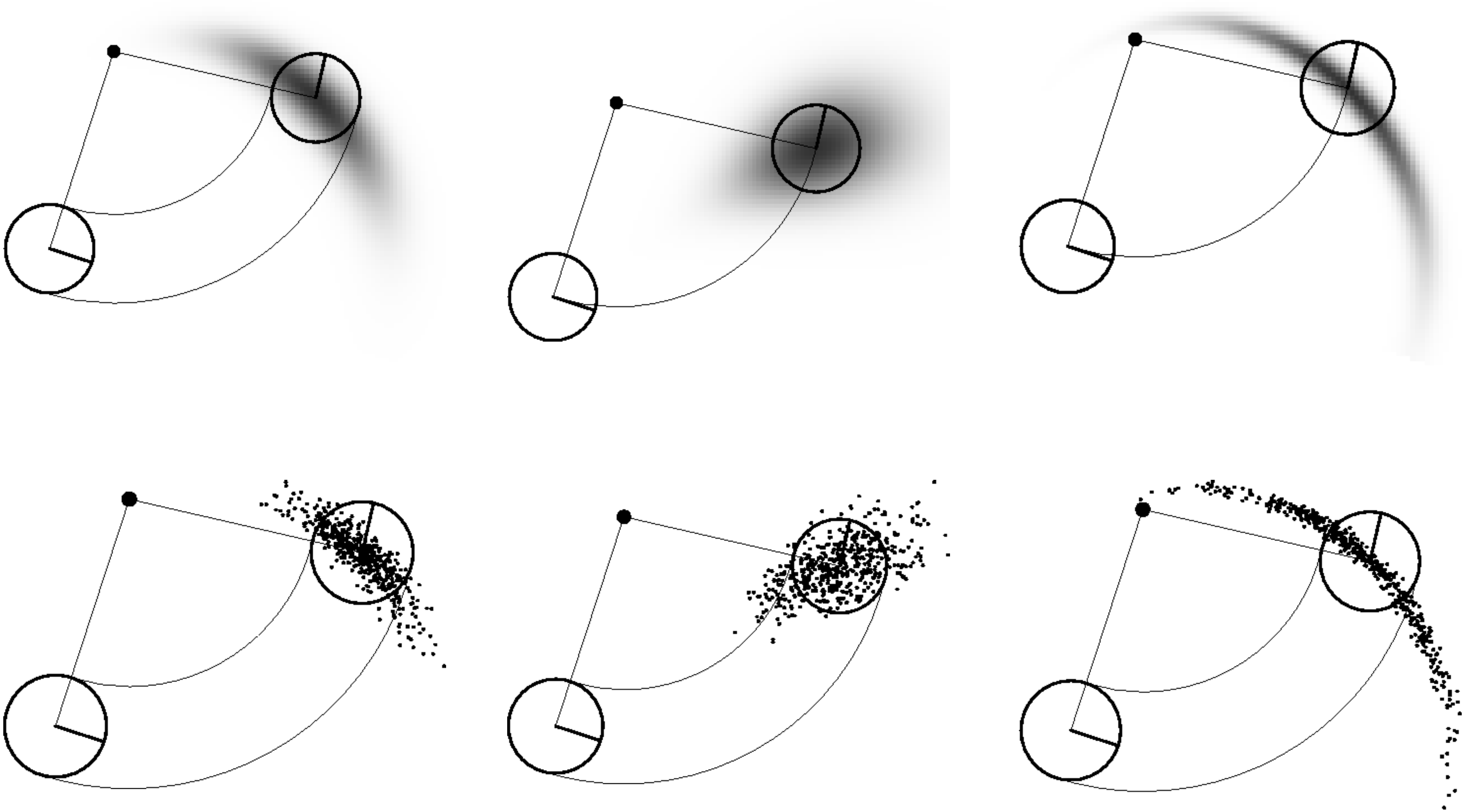
$x' = x - \frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}}\ sin(\theta + \hat{\omega}\Delta t)$

$y' = y + \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}}\ cos(\theta + \hat{\omega}\Delta t)$

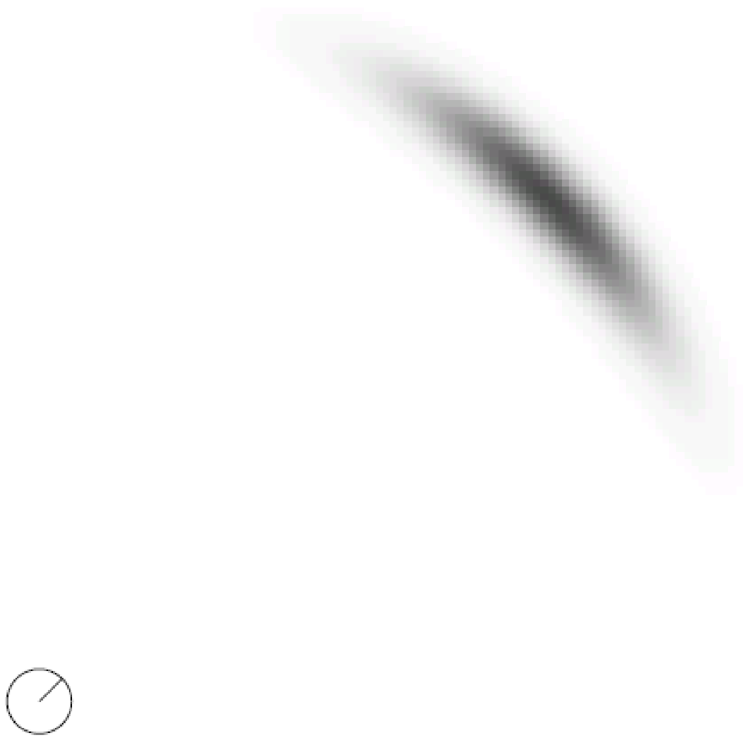
$\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$

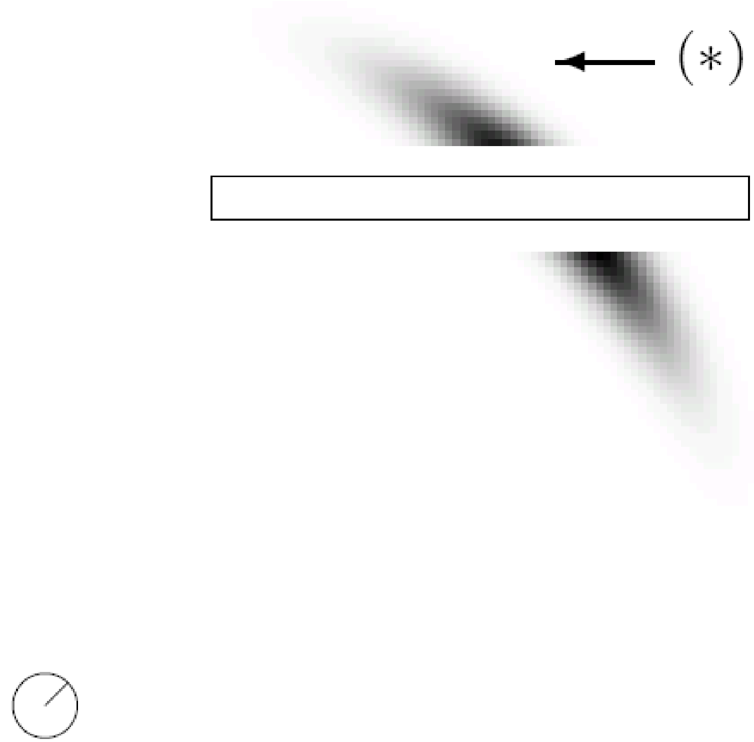
return $x_t = (x', y', \theta')^T$

Examples (Velocity-Based)



Map-Consistent Motion Model





$p(x'|u,x) \neq p(x'|u,x,m)$

Approximation: $p(x'|u,x,m) = \eta\, p(x'|m)\, p(x'|u,x)$

Summary

- We discussed motion models for odometry-based and velocity-based systems
- We discussed ways to calculate the posterior probability $p(x' \mid x, u)$.
- We also described how to sample from $p(x' \mid x, u)$.
- Typically the calculations are done in fixed time intervals Δt.
- In practice, the parameters of the models have to be learned.
- We also discussed how to improve this motion model to take the map into account.

Literature

- Thrun et al. “Probabilistic Robotics”, Chapters 5