(CS 207) Discrete Structures 12 Aug, 2012

Assignment 1

- Due Date: **17 Aug (Fri), 2012**
- DO NOT CHEAT! Copy cases will be handled severely (FR).
- Answer each question on a different sheet of paper.
- If you refer to anything other than class notes (for example, internet, text books etc) do specify the source.

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1. **Akbar and Birbal** (2+2+3+3)

   Here is a slightly modified tale of the king Akbar and his brilliant council member Birbal. Akbar and Birbal decided to play a game. Akbar decided that he will be the one who will think of a natural number and Birbal had to guess the number. Birbal could only guess one number every day. If Birbal guessed correctly, the king promised him 1000 gold coins. If the number is wrong, the game continues. Birbal was said to lose, if the game went on forever.

   (a) What is Birbal's strategy to win?

   (b) If Akbar's number, say $x \in \mathbb{Z}$, then does Birbal have a winning strategy? If so, what is it? (If not – justify).

   (c) Suppose the king chooses an ordered pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ and Birbal is suppose to guess the pair. The guess is correct only if Birbal gets both the numbers right and in that order. Does Birbal have a winning strategy? If so, what is it? (If not – justify).

   (d) Suppose Akbar chooses an arbitrary but finite set $A \subset \mathbb{N}$ of cardinality at most n and Birbal knows n and is suppose to guess a set every time. The guess is correct only if Birbal guesses the exact set A. Does Birbal have a winning strategy? If so, what is it? (If not – justify).

2. Induction (3+2)

   Let $x_0 = 1$ and $x_{i+1} = \left(\sqrt{2}\right)^{x_i}$. Prove that

   (a) $\forall i \in \mathbb{N}, x_{i+1} > x_i$

   (b) $\forall i \in \mathbb{N}, x_i < 2$

3. Let $f : A \to B$. Show the following: (3+3+3+1)

   (a) f is injective iff f has a left inverse

   (b) f is surjective iff f has a right inverse

   (c) f is bijective iff there exists a map $g : B \to A$ such that $f \odot g$ is the identity on B and $g \odot f$ is the identity on A

   Here, left (right) inverse is a function $g : B \to A$ such that $g \odot f(x) = x (f \odot g(x) = x)$. Think of examples in each case.

4. A piece of land (5)

A farmer had 10 children, no two of the same age. After the farmer died, all the brothers decided to split the 100 acre land (into integral parts) between them. After many verbal fights, they decided upon the following strategy: in one round, the eldest of them will propose a division of land. If it is **agreeable** to the **majority**, division will take place and the process will end. Otherwise, all the farmers/brothers will kill the eldest in this round and proceed to the next round. This process will stop when only last two brothers remain. In the last round, the elder will take all the land. If in a round there are n surviving brothers, the **majority** is defined to be $\left(\lceil \frac{n}{2} \rceil\right)$ (i.e. if n is even then majority is $n/2$ and if odd then it is $(n+1)/2$). In any round a farmer is said to find his share *agreeable* if it is strictly more than what he would have got in the next round. The person who proposes the division is assumed to find it agreeable in that round. (His vote is counted.)

What does the eldest of them propose to maximize his share?