CS 207 Discrete Mathematics – 2012-2013

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 11: Catalan numbers

August 28, 2012

Last time

Recap

- Cayley's number: the number of labelled trees is n^{n-2}
- Using generating functions to solve recurrences.

Today

- Doubt solving session.
- Generating functions and recurrences.

Assignment 2, Question 4

Theorem

Let $x_1 \leq x_2 \leq \ldots \leq x_n$. *Prove that if* $\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = 1$ *then* $x_n < 2^{n!}$

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Proof.

- Let the *precision* of a rational $\frac{a}{b}$ be p where, $\frac{a}{b}$ equals $\frac{q}{p}$ after cancellations.

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  $y_k \geq x_k/(n - k + 1)$.

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- And $y_k \leq x_1 x_2 \ldots x_{k-1}$
- $\therefore x_k \leq x_1 x_2 \ldots x_{k-1}(n-k+1)$
- Solving, we get $\forall n > 3 : x_n \leq 2^{n!}$

□

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No! This a is a theorem called Spencer's theorem. It can be derived using another heavy hammer. Those interested may look up on Google. (Beyond the scope for now.)

Cayley's number

Towards the end of the lecture. (Last 5-7 minutes)

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- Note that (l, r) is a bracketed expression where l is a bracketed expression with i terms and r with $n - i$ terms for some i such that $1 \leq i \leq n - 1$. Therefore, the recurrence for $C(n)$:

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How to solve this recurrence? Using generating functions, of course!

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Let $\alpha \in \mathbb{R}^+$. $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, *where* $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}$

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- $C(n) = \frac{(-1)^n}{2^{n+1}}\frac{(-4)^n}{n!}\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-3)$
- $C(n) = \frac{(-1)^{2n}}{2^{n+1}}\frac{(4)^n}{n!}\cdot\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\ldots\cdot(2n-3)\cdot(2n-2)}{2^{n-1}(n-1)!}$
- $C(n) = \frac{1}{\cancel{2^{n+1}}}\frac{\cancel{(4)^n}}{n!}\cdot\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\ldots\cdot(2n-3)\cdot(2n-2)}{\cancel{2^{n-1}}(n-1)!} = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}$

Catalan Number

Theorem (n-th Catalan Number)

If the recurrence for $C(n)$ is given as follows:

$$C(n) = \sum_{i=1}^{n-1} C(i)C(n-i) \qquad \textit{for } n > 1$$

then

$$C(n) = \frac{1}{n}\binom{2n-2}{n-1}$$