CS 207 Discrete Mathematics – 2012-2013

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Combinatorics

Lecture 12: Catalan numbers, derrangements

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Last time

Recap

- Introduction to recurrences and generating functions
- Compute the n-th Catalan number using generating functions

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Theorem (n-th Catalan Number)

If the recurrence for $C(n)$ is given as follows:

$$C(n) = \sum_{i=1}^{n-1} C(i)C(n-i) \qquad \text{for } n > 1$$

then

$$C(n) = \frac{1}{n}\binom{2n-2}{n-1}$$

Today

- Coming up with recurrence relations.
- Computing the number of derrangements
- Exponential generating functions

Find recurrence relations

- [CW] What is the number of different ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines?

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- [CW] What is the number of different ways a convex polygon with $n + 2$ sides can be cut into triangles by connecting vertices with straight lines?
- [CW] What is the number of monotonic paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal?

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$D(0) = 1, D(1) = 0$ by convention.

Closed form for $D(n)$

Theorem

Let $D(n)$ denote the number of derrangements for n elements then

$$D(n) = n!\left(\sum_{i=0}^{n} \frac{(-1)^i}{i!}\right)$$

Proof.

We will prove that RHS has the same recurrence as LHS and RHS matches with LHS for $n = 0, 1$.

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$$(n-1)(f(n-1) + f(n-2))$$
$$= (n-1)\left[(n-1)!\left(\sum_{i=0}^{n-1} \frac{(-1)^i}{i!}\right) + (n-2)!\left(\sum_{i=0}^{n-2} \frac{(-1)^i}{i!}\right)\right]$$

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$= f(n)$

□