CS 207 Discrete Mathematics – 2012-2013

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Mathematical Reasoning and Mathematical Objects

Lecture 2: Induction

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Yesterday

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- Various theorems we proved in class:

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  - Well-ordering principle implies induction.
  - $2^n < n!$
  - $4a^3 + 2b^3 = c^3$ does not have roots over $\mathbb{N}$.

Recap

- What are axioms, propositions, theorems, claims and proofs?
- Various theorems we proved in class:
- The well ordering principle, induction, and strong induction.

You were asked to think about the following two problems:

- Is $2^n < \frac{n}{2}!$?
- For every positive integer n there exists another positive integer k such that n is of the form $9k, 9k+1$, or $9k-1$.

Another proof by induction

Theorem

For any $n \in \mathbb{N}, n \geq 2$ *prove that*

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a slightly stronger induction hypothesis is required

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Proof.

For all $2 \leq i \leq j, i, j \in \mathbb{N}$ let $f(i,j) = \sqrt{i\sqrt{i+1\ldots\sqrt{j}}}$.
We will prove a slightly more general statement:
For all $2 \leq i \leq j, i, j \in \mathbb{N}, f(i,j) < i+1$
This is more general than the theorem statement we wanted to prove.

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For any $n \in \mathbb{N}, n \geq 2$ prove that

$$\sqrt{2\sqrt{3\sqrt{4 \ldots \sqrt{n-1\sqrt{n}}}}} < 3 \Leftarrow \forall 2 \leq i < j, i,j \in \mathbb{N}, f(i,j) < i+1 \qquad (*)$$

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We prove $(*)$ by induction on $j - i$.

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Base case: $j - i = 1$. $f(i,i+1) = \sqrt{i\sqrt{i+1}} < i+1$.

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Induction:

$$\begin{aligned} f(i, j+1) &= \sqrt{i \cdot f(i+1, j+1)} \\ &< \sqrt{i \cdot (i+2)} && \text{(by Induction Hypothesis)} \\ &\leq i+1 && \text{(by AM-GM inequality)} \end{aligned}$$

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Theorem (Bogus, CW2.2)

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$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$$

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???

$\square$