CS 207 Discrete Mathematics – 2012-2013

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Mathematical Reasoning and Mathematical Objects

Lecture 4: Cantor's diagonalisation?

Aug 6, 2012

Last time

Recap

- What are finite and infinite sets?

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- What are functions?

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- What are finite and infinite sets?
- What are functions? What are injective, surjective, and bijective functions?
- Comparing sizes of infinite sets.

Back to infinite sets

We will understand the notion of *size of an infinite set* in a relative sense.

Definition

We say that two sets A, B have the same size if and only if there is a bijection between A and B

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Examples

- Let E be a set of even numbers. There is a bijection between E and $\mathbb{N}$ $f(x) = 2x, f : \mathbb{N} \to E$.

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- There is a bijection $f : \mathbb{Z} \to \mathbb{N}$

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- There is a bijection $f : \mathbb{Z} \to \mathbb{N}$
  $$f(x) = \begin{cases} -2x & \text{if } x \leq 0 \\ 2x - 1 & \text{otherwise} \end{cases}$$

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- There is a bijection $f : \mathbb{Z} \to \mathbb{N}$
- There is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

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- There is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  $f(x, y) = \left(\sum_{i=1}^{x+y} i\right) + y + 1$

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- There is a bijection $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
- Is there a bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$?

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- Is there a bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$?
- Is there a bijection between $\mathbb{R}$ and $\mathbb{N}$?

Finite sets vs infinite sets

On the one hand

- If A is finite then there is no bijection from $A \times A$ to A. Whereas if A is countably infinite

Finite sets vs infinite sets

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- If A is finite then there is no bijection from $A \times A$ to A. Whereas if A is countably infinite, where A is said to be countably infinite if there is a bijection from A to $\mathbb{N}$

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On the other hand

- Today we will see two theorems which prove two properties of infinite sets that they share with finite sets.

Cantor's diagonalisation

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$.

Proof.

Suppose for the sake of contradiction that there is a bijection, say f, between set of all subsets of $\mathbb{N}$.

	0	1	2	3	...
$\emptyset$					
$\{1\}$					
$\{2\}$					
$\{1,2\}$					
$\vdots$					
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$\emptyset$	✗	✗	✗	✗	...
$\{1\}$	✗	✓	✗	✗	...
$\{2\}$	✗	✗	✓	✗	...
$\{1,2\}$	✗	✓	✓	✗	...
:	...	...	...	...	...
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$\{2\}$	✗	✗	✗	✗	...
$\{1,2\}$	✗	✓	✓	✓	...
:	...	...	...	...	...
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$\{1\}$	✗	✗	✗	✗	...
$\{2\}$	✗	✗	✗	✗	...
$\{1,2\}$	✗	✓	✓	✓	...
$\vdots$	...	...	...	...	...
$\vdots$	...	...	...	...	...

The inverted diagonal set does not belong to any of the existing sets! $\square$