CS 207 Discrete Mathematics – 2012-2013

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Mathematical Reasoning and Mathematical Objects

Lecture 5: Schroder-Bernstein

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Last time

Recap

- There is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$

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  Hint: Any point (x, y) such that $x + y = k$ is mapped to an interval of size $k + 1$ which starts at $\frac{k(k+1)}{2}$.

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  Why is this a bijection?
  Hint: Any point (x, y) such that $x + y = k$ is mapped to an interval of size $k + 1$ which starts at $\frac{k(k+1)}{2}$.

  Why injective: If $x + y \neq x' + y'$ then $f(x, y) \neq f(x', y')$.
  If $x + y = x' + y'$ and $f(x, y) = f(x', y')$ then $y = y'$. But if $x + y = x' + y'$ and $y = y'$ then $x = x'$
  Why surjective: Say $\exists k \in \mathbb{N}$ such that k has no inverse. But for any $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\frac{n(n+1)}{2} \leq k \leq \frac{(n+1)(n+2)}{2}$

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- There is no bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$.

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- There is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$
  $f(x, y) = \left(\sum_{i=1}^{x+y} i\right) + y$
- There is no bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$.
  Proof by Cantor's diagonalization. [Cantor, 1891]

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- Another property of sets which holds for both finite and infinite sets. [Schröder-Bernstein Theorem]

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- Another property of sets which holds for both finite and infinite sets. [Schröder-Bernstein Theorem]
- An interesting game and an open problem (If time permits).

Schröder-Bernstein

Theorem

Let A, B be two sets. If there is a injective map g from A to B and another injective map h from B to A then there is a bijection between A, B.

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A toy example:
Say $g : \mathbb{N} \rightarrow \mathbb{N}$, $\mathrm{g}(\mathrm{x}) = \mathrm{x}{+}1$ and $h : \mathbb{N} \rightarrow \mathbb{N}, h(x) = x + 1$.

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Why are g, h injective? Are they bijective?

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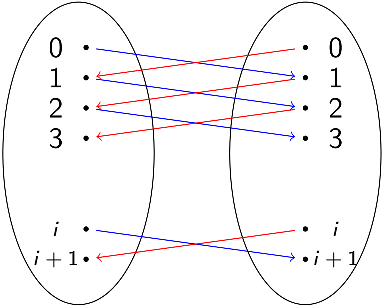

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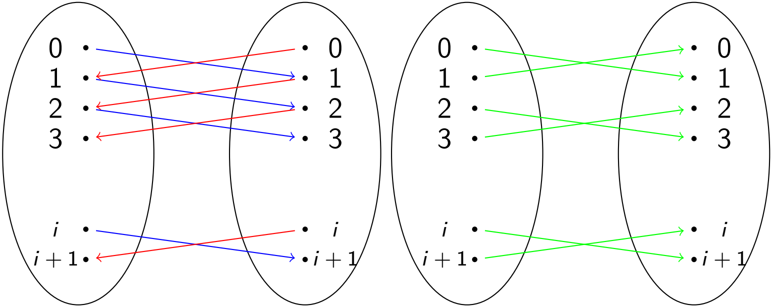

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There are two types of elements in B.

- $B_0 = \{b \in B \mid \exists a \in A \text{ s.t. } g(a) = b\}$
- $B_1 = \{b \in B \mid \nexists a \in A \text{ s.t. } g(a) = b\}$

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An element $b \in B$ be called h-good if $\exists \beta \in B_1, \exists n \in \mathbb{N}$ s.t. $b = (g \odot h)^n \beta$

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What is $(f \odot g)^n$? What does it mean to be h-good?

Schröder-Bernstein

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An element $b \in B$ be called h-good if $\exists \beta \in B_1, \exists n \in \mathbb{N}$ s.t. $b = (g \odot h)^n \beta$
We now define another map from A to B as follows:

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

To finish the proof, we will prove the following lemma about f.

Lemma

The map f defined above is a

Schröder-Bernstein

Lemma

Let $B_1 = \{b \in B \mid \forall a \in A \text{ s.t. } g(a) \neq b\}$, *and*

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

Then f *is injective from* A *to* B

Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.

Schröder-Bernstein

Lemma

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Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.
Case 1 [$g(a), g(a')$ are h-good:] then $f(a) = h^{-1}(a) = f(a') = h^{-1}(a')$. Say $h^{-1}(a) = b_0$. Then we have, $h(b_0) = a$ and $h(b_0) = a'$, i.e. h is not a well-defined functions. This is a contradiction.

Schröder-Bernstein

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Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.

Case 2 [$g(a), g(a')$ are not h-good:] then $f(a) = g(a) = f(a') = g(a')$.

Then we have, $g(a) = g(a')$, i.e. g is not injective. This is a contradiction.

$\square$

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Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.
Case 3[only $g(a)$ is h-good:] We have that $f(a) = f(a')$.
As $g(a)$ is h-good, $f(a) = h^{-1}(a)$. As $g(a')$ is not h-good, $f(a') = g(a')$.

□

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As $g(a)$ is h-good, $f(a) = h^{-1}(a)$. As $g(a')$ is not h-good, $f(a') = g(a')$.

Therefore, $h^{-1}(a) = g(a')$. Call this element b^*.

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Therefore, $h^{-1}(a) = g(a')$. Call this element b^*.

As $g(a') = b^*$, $b^* \notin B_1$. But as $g(a)$ is h-good. Therefore,

$(h \odot g)^{-i}(b^*) \in B_1$ for some $i \in \mathbb{N}$.

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As $g(a') = b^*$, $b^* \notin B_1$. But as $g(a)$ is h-good. Therefore, $(h \odot g)^{-i}(b^*) \in B_1$ for some $i \in \mathbb{N}$.
Assuming $g(a')$ is not h-good, paths walked backwards from b^* lead to B_0.
But Assuming $g(a)$ is h-good, paths walked backwards from b^* lead to B_1.

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But Assuming $g(a)$ is h-good, paths walked backwards from b^* lead to B_1.

Contradiction! □

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$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

Then f *is surjective from* A *to* B

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HW $\square$

Subset Take-away Game – David Gale

I am player 1 and you are player 2. We both have been given a set A. In each round, first I choose one subset of A and then you choose another subset of A. We stick to the following rules:

1. We do not choose the empty set
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First player unable to pick loses the game.

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If $|A| = 1$ then I lose.

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If $|A| = 1$ then I lose. If $|A| = 2$ then you will always win.

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(Source – Mathematics for Computer Science, 2012, by Eric Lehman and F Thomson Leighton)