CS 207 Discrete Mathematics – 2012-2013

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Mathematical Reasoning and Mathematical Objects

Lecture 6: Relations

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Last few classes

Recap

- Proofs, proof methods.

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- Sets and properties of sets

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- Functions, properties of functions
- Infinite sets and properties of infinite sets.

Today

- Relations: generalisations of functions

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- Types and properties of relations

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- Relations: generalisations of functions
- Types and properties of relations
- Representation of functions - directed graphs

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- Relational databases: practical examples of relations.

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  - Is $R(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } a < b\}$ reflexive?
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  - Is $R(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}$ and $a \leq b\}$ symmetric?
  - Is $R(\text{YourClass}) = \{(a, b) \mid a, b \in \text{YourClass}$ and a friend of $b\}$ symmetric?
  - Is $R(\mathcal{P}(S)) = \{(A, B) \mid A \cap B = \emptyset\}$ symmetric?

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  - Is $R(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } a \leq b\}$ transitive?
  - Is $R(\mathbb{N}) = \{(a, b) \mid a \ (mod \ b) \neq 0\}$ transitive?
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  Σ is a finite alphabet. Σ^* are strings of arbitrary length over Σ

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Let $(S, \preceq)$ be a poset. A subset $A \subseteq S$ is called an anti-chain if no two elements of A are related to reach other under $\preceq$.

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- Tum-tum route graphs
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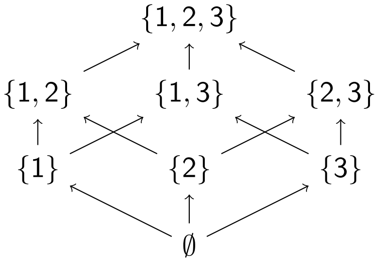

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[CW] What are the chains in this poset?
[CW] What are the anti-chains in this poset?