CS 207 Discrete Mathematics – 2012-2013

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Mathematical Reasoning and Mathematical Objects

Lecture 7: Properties of equivalence relations and partial orders

August 13, 2012

Last time

Recap

- What are relaions?

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- What are different types of functions?

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	reflexive	transitive	symmetric	anti-symmetric
equivalence relation	✓	✓	✓	
partial order	✓	✓		✓

- Representation of partial orders by graphs

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- Recall chains and anti-chains and study properties of partial orders.

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- $[1] = \{1, 5, 9, 13, 17, \ldots\}$

Properties of equivalence relations

Let R be an equivalence relation of A. Let elements of A be x, y, z etc.

Lemma

The following three are equivalent: (a) xRy, (b) $[x] = [y]$, (c) $[x] \cap [y] \neq \emptyset$.

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Proof.

(a) $\Rightarrow$ (b): Say $z \in [x]$. But xRy. As xRy and R is symmetric, yRx. Therefore, yRx, xRz. R is transitive. Therefore, yRz, i.e. $z \in [y]$. This proves that $[x] \subseteq [y]$. The proof of $[y] \subseteq [x]$ is similar.

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(c) $\Rightarrow$ (a): Let $z \in [x] \cap [y]$. Therefore, xRz and yRz. But as R is symmetric, zRy. But R is also transitive. Therefore xRz and zRy imply xRy. □

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Sets $X_1, X_2, \ldots, X_m$ are said to partition a set X if

- $\forall i, j \in \{1, 2, \ldots, m\}, i \neq j : X_i \cap X_j = \emptyset$
- $\forall x \in X, \exists i \in \{1, 2, \ldots, m\} : x \in X_i$

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Let $[x] \neq [y]$ be two distinct equivalence classes of R. From the previous lemma $[x] \cap [y] = \emptyset$.

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Let R be an equivalence relation defined on a set A.

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- *Conversely, given a partition $\{A_i \mid i \in \{1, 2, \ldots, n\}\}$ of A, there is an equivalence relation R_A with equivalence classes $A_1, A_2, \ldots, A_n$.*

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R_A relates (x, y) if they belong to the same part in the partition of A.

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Let $R_A = \{(x, y) \mid \exists i : x, y \in A_i\}$.
R_A is reflexive. If $(x, y) \in R_A$ then even $(y, x) \in R_A$. Finally, if $(x, y) \in R_A$ then $\exists i : x, y \in A_i$. Let that index be called i_0. Now if $(y, z) \in R_A$ then both y, z must be in the same part of the partition. But we know that $y \in A_{i_0}$. Therefore, $z \in A_{i_0}$. Hence, $x, z \in A_{i_0}$ and hence $(x, z) \in R_A$. This proves that R_A is also transitive. □

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Let $(S, \preceq)$ be a poset. A subset $A \subseteq S$ is called an anti-chain if no two elements of A are related to reach other under $\preceq$.

Example

Let $S = \{1, 2, 3\}$. Recall the poset $(S, \subseteq)$.

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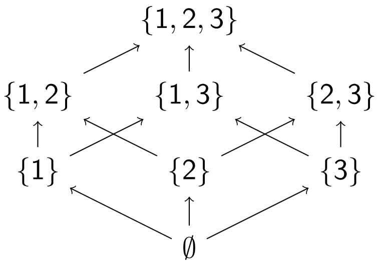

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Let the chain be denoted as $a_1 \preceq a_2 \preceq \ldots \preceq a_m$. Now observe that every element of this chain, must go to different anti-chains.

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Proof.

Let the chain be denoted as $a_1 \preceq a_2 \preceq \ldots \preceq a_m$. Now observe that every element of this chain, must go to different anti-chains. Therefore, there are at least m anti-chains in $(S, \preceq)$. □

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For each element $s \in S$, let C_s be the set of all chains that have s as the maximum element. And define $label(s) := max_{c \in C_s} \{\text{size}(c)\}$.

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[CW] For any $s \in S$, how large can $label(s)$ be?

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□

Subset Take-away problem

Removing supersets $\equiv$ getting rid of some chains.

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Do partial orders help?