CS 207 Discrete Mathematics – 2012-2013

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Combinatorics
Lecture 8: Double counting
August 14, 2012

Course Outline

- Mathematical reasoning and mathematical objects
- Combinatorics

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  - What is a proof? Types of proof methods
  - Induction
  - Sets, relations, functions, partial orders, graphs
- Combinatorics
- Elements of graph theory
- Elements of abstract algebra

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Text:	*Discrete Mathematics and its applictions, by Kenneth Rosen*
	Chapter 2 : $2.1, 2.2, 2.3$, Chapter 8 : $8.1, 8.5, 8.6$
Class notes:	uploaded on Moodle

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- Elements of graph theory
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- Mathematical reasoning and mathematical objects
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  - Approximating sums and products
  - Pigeonhole principle
  - Recurrence relations and generating functions
  - Inclusion-exclusion principle
  - Elements of discrete probability
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  - The rest of pairs, $n^2 - n$ of them, may or may not be put.
  - Therefore, there are 2^{n^2-n} different reflexive relations.

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- How many reflexive relations are there on a set, say A, of size n?
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$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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  - As LHS and RHS are counting the same quantity they must be equal.

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\begin{aligned}
S =& 1+ \quad && 2+\ldots+ \quad && n \\
& n+ \quad && (n-1)+\ldots+ \quad && 1 \\
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$$S = 1 + \quad x + \ldots + \quad x^n \tag{1}$$

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$$S = 1 + x + \ldots + x^n \quad (1)$$
$$xS = x + x^2 + \ldots + x^{n+1} \quad (2)$$

(1)-(2) gives us: $(1-x)S = 1 - x^{n+1}$

  - Therefore, $S = \frac{1-x^{n+1}}{1-x}$.

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  - Each round: player 1 draws a line using a red pen and then player 2 draws a line using a blue pen.

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  - Ramsey proved that a draw is impossible!

Why and how to count?

On various occasions different quantities may become interesting. Some may be easy to count directly. Some may require more thought.

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In this module, we will build some technqiues that will help in counting some quantities which are hard to count.

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Often to count a certain object, we will count some totally different object!

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Proof.

Recall $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

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For the RHS, note that

- Either choose the mango in the basket and select $k-1$ apples from n apples in $\binom{n}{k-1}$ ways.
- Or leave out the mango from the basket and select k apples from n apples in $\binom{n}{k}$ways.

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$\text{Degree}(v) := |\{u \mid (u, v) \in E\}|$

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- On the one hand this number is $\sum_{i=1}^{n} m_i$.

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- On the one hand this number is $\sum_{i=1}^{n} m_i$.
- On the other hand each handshake gives rise to two edges. So if X is the number of handshakes, then the number of edges is $2X$.

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Take back message: Counting the same quantity the number of directed edges in two different ways can be helpful!

Counting the same quantity in different ways

Lemma

Cosnider a class of m students. Every day after class 3 students stay back to clean the classes. At the end of the course, they realise that each pair of students stayed back exactly once. For how many days did the course run?

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$\therefore n = \frac{m(m-1)}{6}$. □