CS 207 Discrete Mathematics – 2012-2013

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Combinatorics

Lecture 9: Counting the same object in two ways

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Last time

Recap

- Counting the same object in two different ways
  - Basic counting
  - $k\binom{n}{k} = n\binom{n-1}{k-1}$
  - $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
  - The number of people who shake hands odd number of times is even.

Today

- How large/small is $n!$? – approximating $n!$ [Stirling's approximation]

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- Counting the number of labelled trees – Cayley's number.

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Theorem (Stirling's approximation)

$e(n/e)^n \leq n! \leq ne(n/e)^n$, i.e. $Q = e(n/e)^n$, and $\alpha = 1/n$

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Proof.

Let $S = \log(n!) = \sum_{i=1}^{n} \log i$. We will bound S using the natural log. From the figure on the board:

$$\sum_{i=1}^{n-1} i \leq \int_1^n \log x \; dx$$

$$\begin{aligned} S &\leq \int_1^n \log x \; dx + \log n \\ &= (x \log x - x)|_1^n + \log n \\ &= n \log n - n + 1 + \log n \end{aligned}$$

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raising both sides to the power of e, we get

$$\begin{aligned} n! &\leq e^{(n+1)\log n - (n-1)} \\ &= n^{n+1}/e^{n-1} \\ &= ne(n/e)^n \end{aligned}$$

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Counting labeled trees – Cayley's number

Recall

- What is a graph?
- What are directed and undirected graphs?
- What is a cycle in a graph?
- What is a tree?

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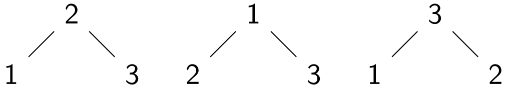
What is a labeled tree?

Example: Labeled trees on 3 vertices



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Doubly rooted trees: labelled trees with two special nodes (both may be the same vertex)

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Suppose we prove that

the number of doubly rooted labeled trees	$=$	total number of functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$
	$=$	n^n

then we are done. $\square$

How many doubly rooted labeled trees on n vertices?

Lemma

There is a bijection between

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From each doubly rooted labeled tree T define a unique functions:
Let u, v be the two roots of T.

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Let u, v be the two roots of T. [CW] How many paths between u, v in T?

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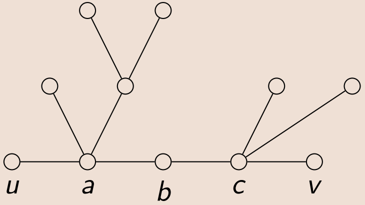

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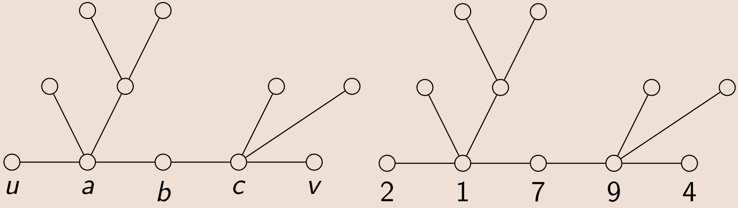

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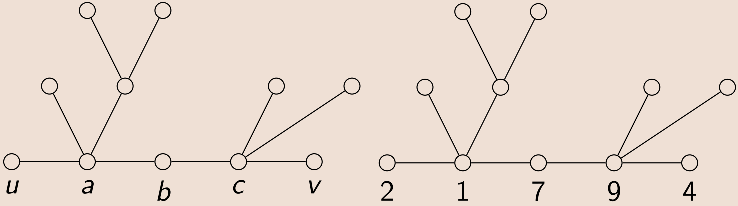

pos(x) :=

distance of x from u

ord(x,S) := order of x in set S.

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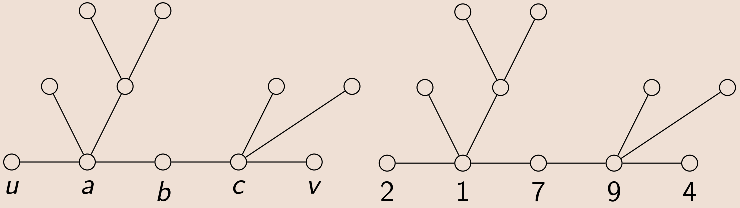

$$\begin{pmatrix} 1 & 2 & 4 & 7 & 9 \\ 2 & 1 & 7 & 9 & 4 \end{pmatrix} \quad f(1) = 2, f(2) = 1, f(4) = 7, f(7) = 9, f(9) = 4$$

$\square$

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Each doubly rooted labelled tree gives rise to a function such that:
The skeleton abels are permutated among themselves and tree nodes are mapped to their parent.

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x is a parent of y if (x, y) is an edge and x is closer to the root than y

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$$f(u) = \begin{cases} j, \text{ where ord(u,S) = pos(j)} & \text{if } u \text{ on skeleton } S \\ \text{parent}(u) & \text{if } u \text{ not on skeleton} \end{cases}$$

□