Tutorial 1

- Solve all questions. Discuss solutions with TAs during TA meeting hours.

1. Prove the following properties of Fibonacci numbers using induction, where Fibonacci numbers are defined as follows:
   $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

   (a) Prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$

   (b) Prove that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$

   (c) Prove that $F_1 + F_3 + F_5 + \ldots + F_{2n+1} = F_{2n+2}$, and $1 + F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1}$

2. Using the well-ordering principle prove the following statement: Any $n \in \mathbb{Z}^+$, if $n > 1$ then it can be factored as a product of primes.

3. Find bugs in the following proof:

   Claim 0.1 (bogus)**.** *Every Fibonacci number is even.*

   Faulty proof. The proof is by using the well-ordering principle (WOP). Suppose there are odd Fibonacci numbers. Let C be the set of all such numbers. By the WOP, we know that there is a minimum number m such that F_m is odd. As F_0 is even, $m > 0$. Since m is the minimum in C, we know that F_{m-1}, F_{m-2} are both even. But by the definition of $F_m = F_{m-1} + F_{m-2}$, F_m is a sum of two even numbers. Therefore, it must be even. Therefore, our assumption that there are odd Fibonacci numbers must be wrong, thereby proving the claim! $\square$

4. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show (using induction) that there is a car which can complete a lap by collecting gas from the other cars on its way around.

5. There is a building with n floors. You are given two identical eggs. There is some floor $h \leq n$ such that if the eggs are thrown from h then they break and if the eggs are thrown from another floor $k < h$ then they do not break. Give an optimal strategy to compute h.