CS 207 Discrete Mathematics – 2013-2014

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Mathematical Reasoning and Mathematical Objects

Lecture 1: What is a proof?

July 18, 2013

Credit Structure

Course credit structure

quizzes	25%
mid-sem	35%
end-sem	40%

Office hours:	11:00am to 1:00pm (Wednesday)
Problem solving session:	1 hour per week (To be announced)

Important Announcements

Quiz 1: August 28, 2013 , Wednesday, 8:30am to 9:30am

Quiz 2: September 4, 2013 , Wednesday, 8:30am to 9:30am

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Next class:	July 29, 2013

Course Outline

- Mathematical reasoning and mathematical objects
- Combinatorics
- Elements of graph theory
- Elements of abstract algebra

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  - What is a proof? Types of proof methods
  - Induction
  - Sets, relations, functions, partial orders, graphs
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Text: *Discrete Mathematics and its applictions, by Kenneth Rosen*
Chapter 2 : 2.1, 2.2, 2.3, Chapter 8 : 8.1, 8.5, 8.6
Class notes: will be uploaded on Moodle

- Combinatorics
- Elements of graph theory
- Elements of abstract algebra

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A statement that is either true or false.

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  $\forall$: for all,
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  $\in, \notin$: contained in, and not contained in
  $\mathbb{N}$: the set of natural numbers,
  $\mathbb{Z}$: the set of integers,
  $\mathbb{Q}$: the set of rationals,
  $\mathbb{Z}^+$: the set of positive integers,
  $\mathbb{R}$: the set of reals

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- $2 + 2 = 4$, every odd number is a prime, there are no even primes other than 2;
- $\forall a, b \in \mathbb{N}, \exists c \in \mathbb{N} : a^2 + b^2 = c$;
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- $\forall a, b \in \mathbb{N}, \exists c \in \mathbb{Z} : a^2 - b^2 = c$;

It is not always easy to tell whether a proposition is true or false.

Theorems and proofs

Theorem

If $0 \leq x \leq 2$, *then* $-x^3 + 4x + 1 > 0$

Theorems and proofs

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(scratchpad)

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(scratchpad)

Proof.

As $-x^3 + 4x = x(4 - x^2)$, which is in fact $x(2 - x)(2 + x)$, the quantity is ~~positive~~ non-negative for $0 \leq x \leq 2$. Adding 1 to a non-negative quantity makes it positive. Therefore, the above theorem. □

Theorems and Proofs

Given: a number $n \in \mathbb{N}$

Check: Is n prime?

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for $i = 2$ to $\sqrt{n}$ **do**
  if $i \mid n$ **then**
    output "no"
  end if
end for

Theorems and Proofs

Given: a number $n \in \mathbb{N}$
Check: Is n prime?

for $i = 2$ to $\sqrt{n}$ **do**
  if $i|n$ **then**
    output "no"
  end if
end for

Why is this algorithm correct?
Is there a number $n \in \mathbb{N}$ s.t
$\forall i : i \in \{2, 3, \ldots, \sqrt{n}\}\ i \nmid n$,
but $\exists j > \sqrt{n}$ s.t. $j|n$?

Is there a composite $n \in \mathbb{N}$ s.t. all its prime factors are greater than $\sqrt{n}$?

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Theorem

If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$

Proof.

As n is a composite, $\exists x, y \in \mathbb{N}, x, y < n : n = xy$. If $x > \sqrt{n}$ and $y > \sqrt{n}$ then $xy > n$. Therefore, one of x or y is less than or equal to $\sqrt{n}$. Say x is smaller than $\sqrt{n}$. It is either a composite or a prime. If it is a prime, then we are done. Else, it has prime factorization (axiom: unique factorization in $\mathbb{N}$) and again, we are done. □

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Axioms

Euclid in 300BC invented the method of axioms-and-proofs.

Using only a handful of axioms called Zermelo-Fraenkel and Choice (ZFC) and a few rules of deductions the entire mathematics can be deduced!

Proving theorems starting from ZFC alone is tedious. 20,000+ lines proof for $2 + 2 = 4$

We will assume a whole lot of axioms to prove theorems: all familiar facts from high school math.

Class problems

- (CW1.1) Prove that for any $n \in \mathbb{N}$, $n(n^2 - 1)(n + 2)$ is divisible by 4. (what about divisible by 8?)
- (CW1.2) Prove that for any $n \in \mathbb{N}$, $2^n < (n + 2)!$

Bogus proofs

Theorem (Bogus)

$1/8 > 1/4$

Proof.

$$3 > 2$$
$$3 \log_{10}(1/2) > 2 \log_{10}(1/2)$$
$$\log_{10}(1/2)^3 > \log_{10}(1/2)^2$$
$$(1/2)^3 > (1/2)^2$$

Another bogus proof

Theorem

For all non-negative numbers a, b $\frac{a+b}{2} \geq \sqrt{ab}$

Proof.

$$\frac{a+b}{2} \geq^{?} \sqrt{ab}$$

$$a+b \geq^{?} 2\sqrt{ab}$$

$$a^2+2ab+b^2 \geq^{?} 4ab$$

$$a^2-2ab+b^2 \geq^{?} 0$$

$$(a-b)^2 \geq 0$$

□

Proof Methods

Proof by contrapositive

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Definition (Contrapozitive)

The contrapositive of "if P then Q" is "if $\neg Q$ then $\neg P$"

Proof by contrapositive

Theorem

If r is irrational then $\sqrt{r}$ is also irrational.
If $\sqrt{r}$ is rational then r is rational.

Proof.

Suppose $\sqrt{r}$ is rational. Then $\sqrt{r} = p/q$ for $p, q \in \mathbb{Z}$. Therefore, $r = p^2/q^2$. □

Proof by contradiction

Theorem

$\sqrt{2}$ *is irrational.*

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Proof.

Suppose not. Then there exists $p, q \in \mathbb{Z}$ such that $\sqrt{2} = p/q$, where p, q do not have any common divisors. Therefore, $2q^2 = p^2$, i.e. p^2 is even.

Proof by contradiction

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Suppose not. Then there exists $p, q \in \mathbb{Z}$ such that $\sqrt{2} = p/q$, where p, q do not have any common divisors. Therefore, $2q^2 = p^2$, i.e. p^2 is even. (CW2.1) If p^2 is even, then p is even.

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Suppose not. Then there exists $p, q \in \mathbb{Z}$ such that $\sqrt{2} = p/q$, where p, q do not have any common divisors. Therefore, $2q^2 = p^2$, i.e. p^2 is even.
If p^2 is even, then p is even. (why?)
Suppose not, i..e p^2 is even but p is not. Then $p = 2k + 1$ for some integer k. $p^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. As $4(k^2 + k)$ is even, $4k^2 + 4k + 1$ is odd, which is a contradiction.

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(CW2.2) Prove that there are infinitely many primes.

Well-ordering principle and Induction

Axiom (WOP)

Every nonempty set of non-negative integers has a smallest element.

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Every nonempty set of non-negative integers has a smallest element.

Axiom (Induction)

Let $P(n)$ be a property of non-negative integers. If

1. *$P(0)$ is true (Base case)*
2. *for all $n \geq 0$, $P(n) \Rightarrow P(n+1)$ (Induction step)*

then $P(n)$ is true for for all $n \in \mathbb{N}$.

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Axiom (Strong Induction)

Let $P(n)$ be a property of non-negative integers. If

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2. *$[\forall k \in \{0, 1, \ldots, n\} : P(k)] \Rightarrow P(n+1)$ (Induction step)*

then $P(n)$ is true for for all $n \in \mathbb{N}$.

WOP $\Rightarrow$ Induction

Theorem

Well-ordering principle implies Induction

Proof.

Let $P(0)$ be true and for each $n \geq 0$, let $P(n) \Rightarrow P(n+1)$.
Let us assume for the sake of contradiction that $P(n)$ is not true for all positive integers.
Let $C = \{i \mid P(i) \text{ is false}\}$. As C is non-empty and non-negative integers C has a smallest element (due to WOP), say i_0.
Now, $i_0 \neq 0$. Also $P(i_0 - 1)$ is true, as $i_0 - 1$ is not in C. But $P(i_0 - 1) \Rightarrow P(i_0)$, which is a contradiction. $\square$

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Theorem

WOP $\Leftrightarrow$ Induction $\Leftrightarrow$ Strong Induction [HW]

Using Induction to prove theorems

Theorem

$2^n \leq (n+1)!$

Proof.

Base case ($n = 0$): $2^0 = 1 = 1!$

Using Induction to prove theorems

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Proof.

Base case ($n = 0$): $2^0 = 1 = 1!$
Induction hypothesis: $2^n \leq (n+1)!$.

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\leq 2 \cdot (n+1)! \text{ (by indiction hypothesis)} \\ &\leq (n+2) \cdot (n+1)! \\ &\leq (n+2)! \end{aligned}$$

□

Using Well-ordering principle to prove theorems

Here is a slightly non-trivial example:

Theorem

The following equation does not have any solutions over $\mathbb{N}$:
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Observe that C^3 is even. Therefore, C is even. Say $C = 2\gamma$.
Therefore, $4A^3 + 2B^3 = 8\gamma^3$, i.e. $2A^3 + B^3 = 4\gamma^3$.

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And, now we can repeat the argument with respect to A.
Therefore, if (A, B, C) is a solution then so is (α, β, γ).

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But $\beta < B$, which is a contradiction.

Using Well-ordering principle to prove theorems

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Theorem

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It is not always as easy to prove such theorems.

Conjecture (Euler, 1769)

There are no positive integer solutions over $\mathbb{Z}$ *to the equation:*

$$a^4 + b^4 + c^4 = d^4$$

Integer values for a, b, c, d that do satisfy this equation were first discovered in 1986.
It took more two hundred years to prove it.

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Mathematical Reasoning and Mathematical Objects

Lecture 2: Well-ordering principle and Induction

July 29, 2013

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July 29, 2013

Last time

Recap

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- Various theorems we proved in class:

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  - $\sqrt{2}$ is irrational.

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  - $2^n < n!$

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- What are axioms, propositions, theorems, claims and proofs?
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  - If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$.
  - If r is irrational then $\sqrt{r}$ is irrational.
  - $\sqrt{2}$ is irrational.
  - Well-ordering principle implies induction.
  - $2^n < n!$
- The well ordering principle, induction, and strong induction.

Recap

- What are axioms, propositions, theorems, claims and proofs?
- Various theorems we proved in class:
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You were asked to think about the following problem:

- Is $2^n < \frac{n}{2}!$?

Recap

- What are axioms, propositions, theorems, claims and proofs?
- Various theorems we proved in class:
- The well ordering principle, induction, and strong induction.

You were asked to think about the following problem:

- Is $2^n < \frac{n}{2}!$?
- Try to also think about the following: (CW2.1)
  For every positive integer n there exists another positive integer k such that n is of the form $9k, 9k+1$, or $9k-1$.

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Now, B^3 is even and so is B. Say $B = 2\beta$. $\therefore 2A^3 + 8\beta^3 = 4\gamma^3$.
And, now we can repeat the argument with respect to A.
Therefore, if (A, B, C) is a solution then so is (α, β, γ).
But $\beta < B$, which is a contradiction.

Using Well-ordering principle to prove theorems

Here is a slightly non-trivial example:

Theorem

The following equation does not have any solutions over $\mathbb{N}$:
$4a^3 + 2b^3 = c^3$

It is not always as easy to prove such theorems.

Conjecture (Euler, 1769)

There are no positive integer solutions over $\mathbb{Z}$ *to the equation:*

$$a^4 + b^4 + c^4 = d^4$$

Integer values for a, b, c, d that do satisfy this equation were first discovered in 1986.
It took more two hundred years to prove it.

A proof by induction

Theorem

For any $n \in \mathbb{N}, n \geq 2$ *prove that*

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{n-1\sqrt{n}}}}} < 3$$

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Proof.

For all $2 \leq i \leq j, i, j \in \mathbb{N}$ let $f(i,j) = \sqrt{i\sqrt{i+1\ldots\sqrt{j}}}$.
We will prove a slightly more general statement:
For all $2 \leq i \leq j, i, j \in \mathbb{N}, f(i,j) < i+1$
This is more general than the theorem statement we wanted to prove.

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For any $n \in \mathbb{N}, n \geq 2$ *prove that*

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We prove $(*)$ by induction on $j - i$.

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Base case: $j - i = 1$. $f(i, i+1) = \sqrt{i\sqrt{i+1}} < i+1$.

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Base case: $j - i = 1$. $f(i, i+1) = \sqrt{i\sqrt{i+1}} < i+1$.
Induction:

$$\begin{aligned} f(i,j+1) &= \sqrt{i \cdot f(i+1,j+1)} \\ &< \sqrt{i \cdot (i+2)} && \text{(by Induction Hypothesis)} \\ &\leq i+1 && \text{(by AM-GM inequality)} \end{aligned}$$

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Theorem (Bogus, CW2.2)

$a \in \mathbb{R}$, $a > 0$. *Then*, $\forall n \in \mathbb{N}$, $a^n = 1$.

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???

CS 207 Discrete Mathematics – 2012-2013

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Mathematical Reasoning and Mathematical Objects

Lecture 3: Mathematical structures

Aug 01, 2012

Last time

Recap

- The well-ordering principle.

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- The well-ordering principle.
- The principle of induction: we proved that $\forall i, j \in \mathbb{N},\ f(i,j) < i+1$, where $f(i,j) = \sqrt{i\sqrt{i+1 \ldots \sqrt{j-1\sqrt{j}}}}$.

Mathematical Structures

sets, functions, relations, graphs ...

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Of course, barber could neither shave himself and nor could he not shave himself!
This is called a paradox.

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Cantor was the first person to define sets formally – finite sets as well as infinite sets, and prove important properties related to sets.
Let P be a property then he said any collection of objects which satisfy property P is a set, i.e.
$S = \{x \mid P(x)\}$.

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$A = \{X \mid X \notin X\}$
Now if $A \in A$ then $A \notin A$ and if $A \notin A$ then $A \in A$!

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(CW) Can you come up with a set that contains itself?

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How to get around this paradox?

Definition

Start with a few objects *defined* as sets. Now if A is a set and P is a property, then $S = \{x \in A \mid P(x)\}$ is also a set.

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Why does this definition get rid of Russell's paradox?

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  - ($S \in S$:) from the definition of S, $S \in A$ and $S \notin S$, which is a contradiction.
  - ($S \notin S$:) from the definition, either $S \notin A$ or $S \in S$. But we have assumed that $S \notin S$, therefore it must mean $S \notin A$. There is no contradiction!

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How to get around Barber's paradox? (CW)

Examples and properties

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- Similarly, union, intersection, symmetric difference are defined as:
  $A \cup B = \{x \mid a \in A \text{ or } x \in B\}$
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  Example: Let $A = \{a, b\}$ then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

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- Given two infinite sets, can we talk about one being *bigger* than the other? If so, how?

Functions

Definition

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$f = \{(a, b) \mid a \in A, b \in B\}$ with an additional properties that if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$ and for every $a \in A$ there a $b \in B$ such that $(a, b) \in f$.

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- Injective function, one-to-one: A function $f : A \to B$ is said to be injective if $\forall x, y \in A$ if $f(x) = f(y)$ then $x = y$.
- Surjective function, onto: A function $f : A \to B$ is said to be surjective if $\forall x \in B \ \exists a \in A$ such that $f(a) = x$.
- Bijective function: A function is said to be bijective if it is surjective and injective.
  - Is $f : \mathbb{Z} \to \mathbb{Z}$, defined as $f(n) = n + 1$ bijective?
  - Is $f : \mathbb{Z} \to \mathbb{Z}$, defined as $f(n) = n^2$ bijective?
  - What about $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = 10x - 7$?

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Back to infinite sets

We will understand the notion of *size of an infinite set* in a relative sense.

Definition

We say that two sets A, B have the same size if and only if there is a bijection between A and B

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- Let E be a set of even numbers. There is a bijection between E and $\mathbb{N}$

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Examples

- Let E be a set of even numbers. There is a bijection between E and $\mathbb{N}$ $f(x) = 2x, f : \mathbb{N} \to E$.

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Examples

- Let E be a set of even numbers. There is a bijection between E and $\mathbb{N}$
- There is a bijection $f : \mathbb{Z} \to \mathbb{N}$

$$f(x) = \begin{cases} -2x & \text{if } x \leq 0 \\ 2x - 1 & \text{otherwise} \end{cases}$$

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- Let E be a set of even numbers. There is a bijection between E and $\mathbb{N}$
- There is a bijection $f : \mathbb{Z} \to \mathbb{N}$
- There is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

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- There is a bijection $f : \mathbb{Z} \to \mathbb{N}$
- There is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
  $f(x, y) = \left(\sum_{i=1}^{x+y} i\right) + y + 1$

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- There is a bijection $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

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  CW

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- There is a bijection $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
- Is there a bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$?

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- Is there a bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$?
- Is there a bijection between $\mathbb{R}$ and $\mathbb{N}$?

CS 207 Discrete Mathematics – 2013-2014

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Mathematical Reasoning and Mathematical Objects

Lecture 4: Properties of infinite sets

Aug 1, 2013

Last time

Recap

- How to define sets?
- What are finite and infinite sets?

Recap

- How to define sets?
- What are finite and infinite sets?
- What are functions?

Recap

- How to define sets?
- What are finite and infinite sets?
- What are functions? What are injective, surjective, and bijective functions?

Recap

- How to define sets?
- What are finite and infinite sets?
- What are functions? What are injective, surjective, and bijective functions?
- Comparing sizes of infinite sets.

Size of infinite sets

We will understand the notion of *size of an infinite set* in a relative sense.

Definition

We say that two sets A, B have the same size if and only if there is a bijection between A and B

Size of infinite sets

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Examples

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- There is a bijection $f : \mathbb{Z} \to \mathbb{N}$
- There is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
- There is a bijection $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

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Examples

- Let E be a set of even numbers. There is a bijection between E and $\mathbb{N}$
- There is a bijection $f : \mathbb{Z} \to \mathbb{N}$
- There is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
- There is a bijection $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
- Is there a bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$?
- Is there a bijection between $\mathbb{R}$ and $\mathbb{N}$?

Finite sets vs infinite sets

On the one hand

- If A is finite then there is no bijection from $A \times A$ to A. Whereas if A is countably infinite

Finite sets vs infinite sets

On the one hand

- If A is finite then there is no bijection from $A \times A$ to A. Whereas if A is countably infinite, where A is said to be countably infinite if there is a bijection from A to $\mathbb{N}$

Finite sets vs infinite sets

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- Today we will see two theorems which prove two properties of infinite sets that they share with finite sets.

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Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$.

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Theorem (Cantor, 1891)

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Theorem

Let A, B be two sets. If there is a injective map g from A to B and another injective map h from B to A then there is a bijection between A, B.

Cantor's diagonalisation

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$.

Proof.

Suppose for the sake of contradiction that there is a bijection, say f, between set of all subsets of $\mathbb{N}$.

	0	1	2	3	...
$\emptyset$					
$\{1\}$					
$\{2\}$					
$\{1,2\}$					
:					
:					

Cantor's diagonalisation

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	0	1	2	3	...
$\emptyset$	✗	✗	✗	✗	...
$\{1\}$	✗	✓	✗	✗	...
$\{2\}$	✗	✗	✓	✗	...
$\{1,2\}$	✗	✓	✓	✗	...
⋮	...	...	...	...	...
⋮	...	...	...	...	...

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$\emptyset$	✓	✗	✗	✗	...
$\{1\}$	✗	✗	✗	✗	...
$\{2\}$	✗	✗	✗	✗	...
$\{1,2\}$	✗	✓	✓	✓	...
:	...	...	...	...	...
:	...	...	...	...	...

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$\{1,2\}$	✗	✓	✓	✓	...
:	...	...	...	...	...
:	...	...	...	...	...

The inverted diagonal set does not belong to any of the existing sets! □

Schröder-Bernstein

Theorem

Let A, B be two sets. If there is a injective map g from A to B and another injective map h from B to A then there is a bijection between A, B.

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A toy example:
Say $g : \mathbb{N} \to \mathbb{N}$, $g(x) = x+1$ and $h : \mathbb{N} \to \mathbb{N}, h(x) = x + 1$.

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Say $g : \mathbb{N} \to \mathbb{N}$, $g(x) = x+1$ and $h : \mathbb{N} \to \mathbb{N}, h(x) = x + 1$.
Why are g, h injective? Are they bijective?

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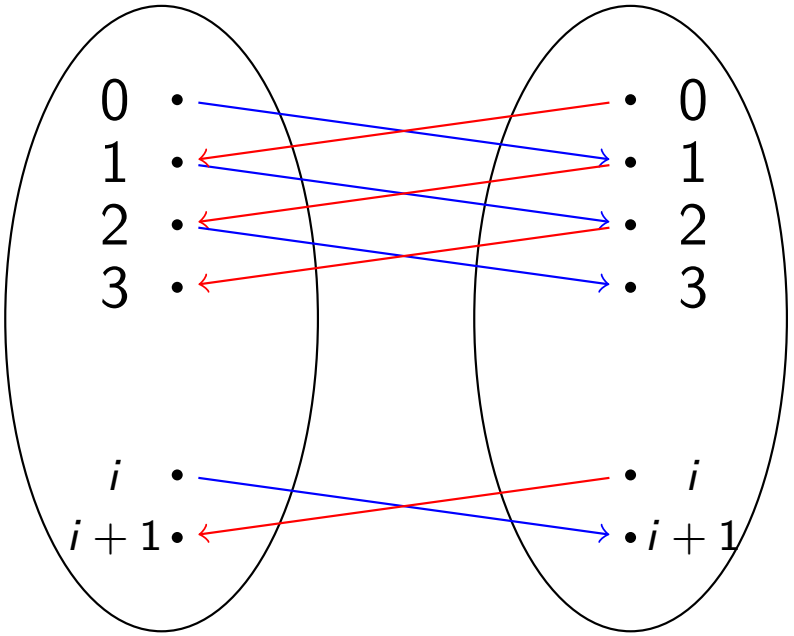

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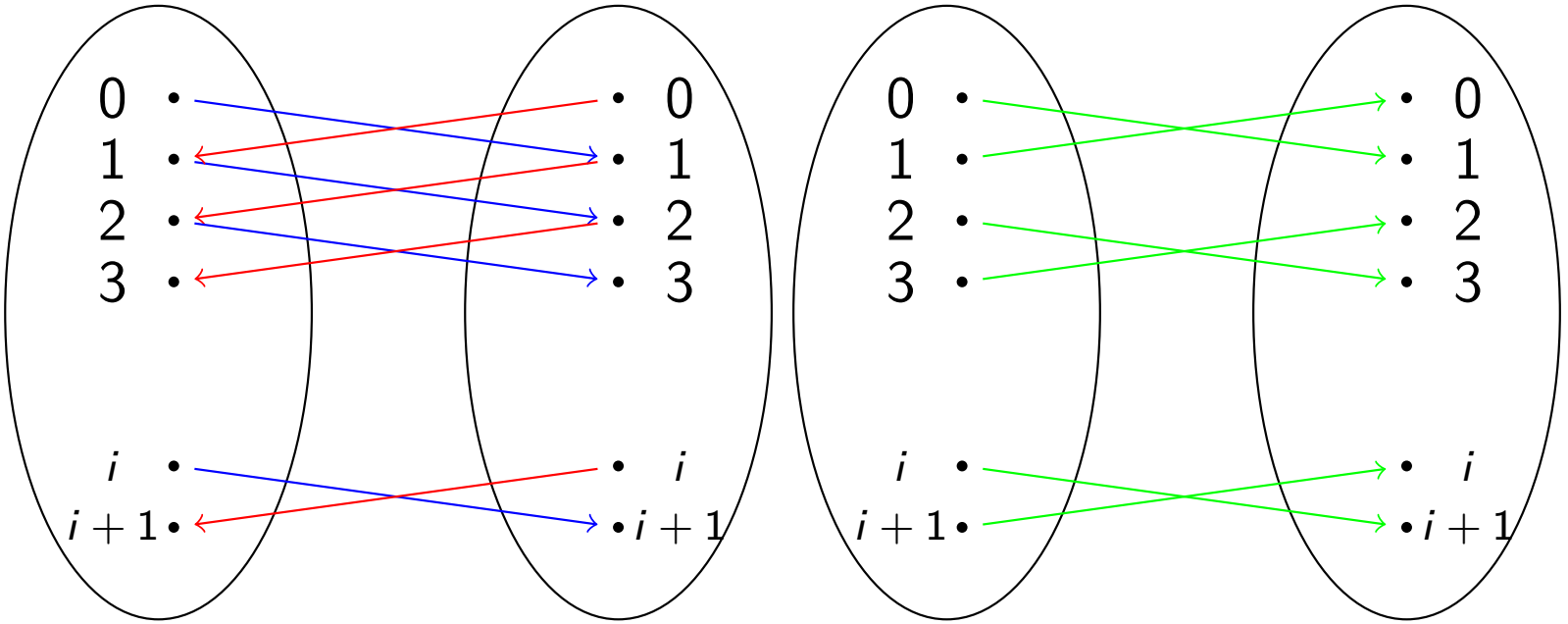

CS 207 Discrete Mathematics – 2013-2014

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Mathematical Reasoning and Mathematical Objects

Lecture 5: Schroder-Bernstein

Aug 5, 2013

Recap

- There is no bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$.

Recap

- There is no bijection between $\mathbb{N}$ and set of all subsets of $\mathbb{N}$. Proof by Cantor's diagonalization. [Cantor, 1891]

Today

- Another property of sets which holds for both finite and infinite sets. [Schröder-Bernstein Theorem]

Today

- Another property of sets which holds for both finite and infinite sets. [Schröder-Bernstein Theorem]
- An interesting game and an open problem (If time permits).

Schröder-Bernstein

Theorem

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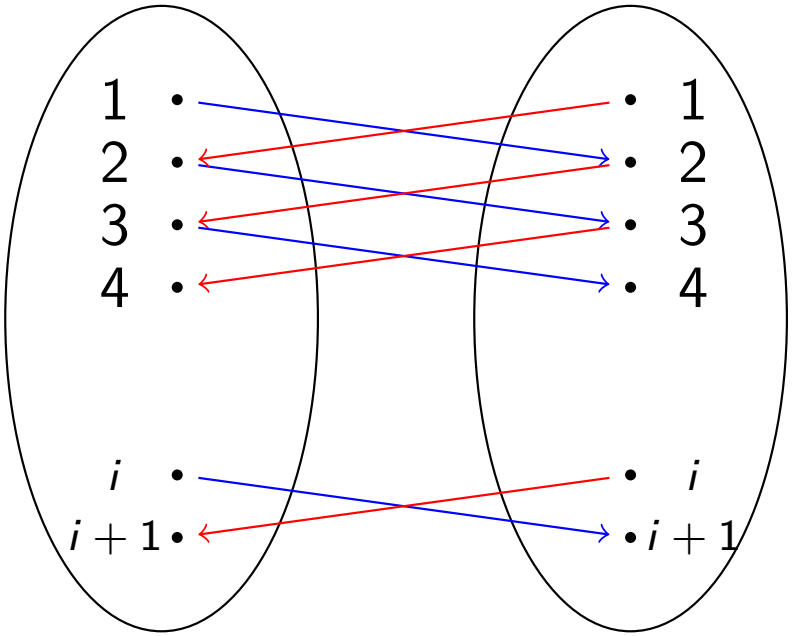

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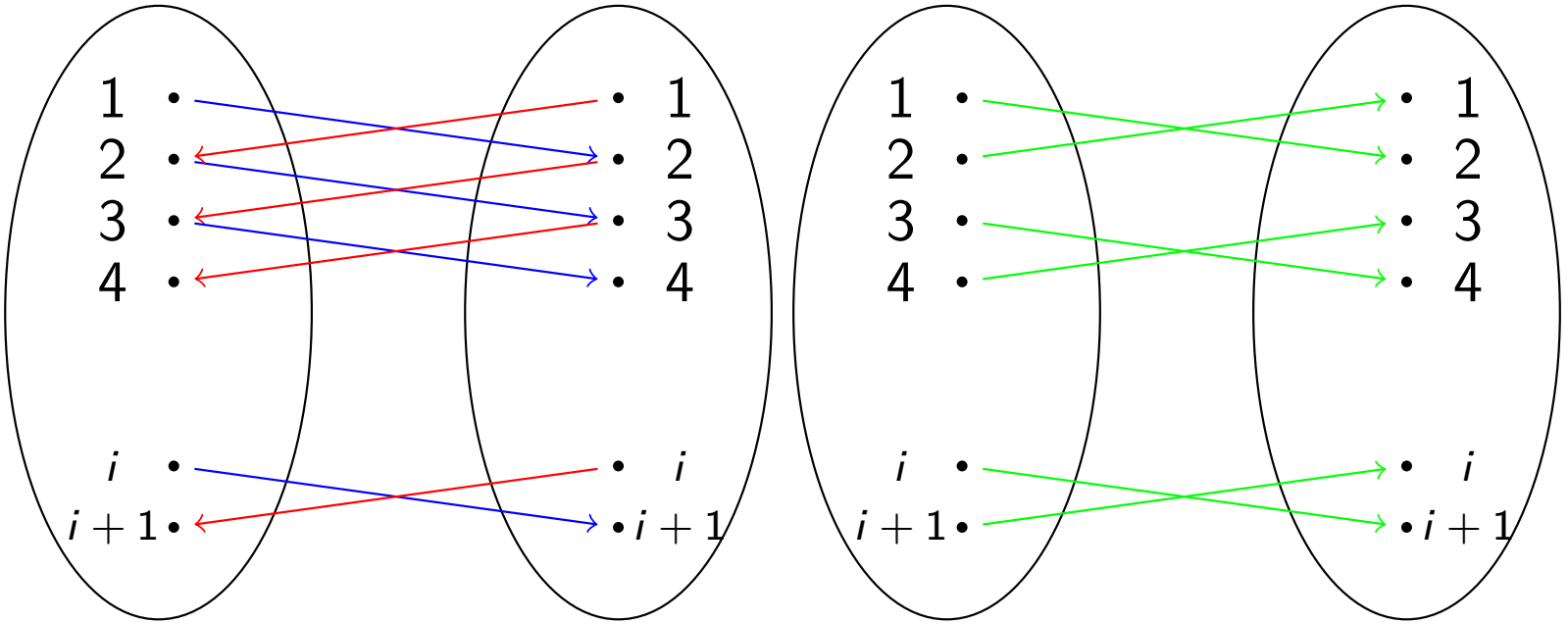

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Schröder-Bernstein

Theorem

Let A, B be two sets. If there is a injective map g from A to B and another injective map h from B to A then there is a bijection between A, B.

There are two types of elements in B.

- $B_0 = \{b \in B \mid \exists a \in A \text{ s.t. } g(a) = b\}$
- $B_1 = \{b \in B \mid \nexists a \in A \text{ s.t. } g(a) = b\}$

Schröder-Bernstein

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An element $b \in B$ be called h-good if $\exists \beta \in B_1, \exists n \in \mathbb{N}$ s.t. $b = (g \odot h)^n \beta$

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An element $b \in B$ be called h-good if $\exists \beta \in B_1, \exists n \in \mathbb{N}$ s.t. $b = (g \odot h)^n \beta$
What is $(f \odot g)^n$? What does it mean to be h-good?

Schröder-Bernstein

Theorem

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There are two types of elements in B.

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An element $b \in B$ be called h-good if $\exists \beta \in B_1, \exists n \in \mathbb{N}$ s.t. $b = (g \odot h)^n \beta$

We now define another map from A to B as follows:

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

To finish the proof, we will prove the following lemma about f.

Lemma

The map f defined above is a bijection.

Schröder-Bernstein

Lemma

Let $B_1 = \{b \in B \mid \forall a \in A \text{ s.t. } g(a) \neq b\}$, *and*

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is h-good} \\ g(a) & \text{otherwise} \end{cases}$$

Then f *is injective from* A *to* B

Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.

Schröder-Bernstein

Lemma

Let $B_1 = \{b \in B \mid \forall a \in A \text{ s.t. } g(a) \neq b\}$, *and*

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

Then f *is injective from* A *to* B

Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.
Case 1 [$g(a), g(a')$ are h-good:] then $f(a) = h^{-1}(a) = f(a') = h^{-1}(a')$.
Say $h^{-1}(a) = b_0$. Then we have, $h(b_0) = a$ and $h(b_0) = a'$, i.e. h is not a well-defined functions. This is a contradiction.

Schröder-Bernstein

Lemma

Let $B_1 = \{b \in B \mid \forall a \in A \text{ s.t. } g(a) \neq b\}$, *and*

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

Then f *is injective from* A *to* B

Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.
Case 2 [$g(a), g(a')$ are not h-good:] then $f(a) = g(a) = f(a') = g(a')$.
Then we have, $g(a) = g(a')$, i.e. g is not injective. This is a contradiction.

□

Schröder-Bernstein

Lemma

Let $B_1 = \{b \in B \mid \forall a \in A \text{ s.t. } g(a) \neq b\}$, *and*

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

Then f *is injective from* A *to* B

Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.

Case 3[only $g(a)$ is h-good:] We have that $f(a) = f(a')$.

As $g(a)$ is h-good, $f(a) = h^{-1}(a)$. As $g(a')$ is not h-good, $f(a') = g(a')$.

□

Schröder-Bernstein

Lemma

Let $B_1 = \{b \in B \mid \forall a \in A \text{ s.t. } g(a) \neq b\}$, *and*

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

Then f *is injective from* A *to* B

Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.
Case 3[only $g(a)$ is h-good:] We have that $f(a) = f(a')$.
As $g(a)$ is h-good, $f(a) = h^{-1}(a)$. As $g(a')$ is not h-good, $f(a') = g(a')$.
Therefore, $h^{-1}(a) = g(a')$. Call this element b^*.

□

Schröder-Bernstein

Lemma

Let $B_1 = \{b \in B \mid \forall a \in A \text{ s.t. } g(a) \neq b\}$, *and*

$$f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$$

Then f *is injective from* A *to* B

Proof.

Suppose (for the sake of contradiction) $a \neq a'$ and $f(a) = f(a')$.

Case 3[only $g(a)$ is h-good:] We have that $f(a) = f(a')$.

As $g(a)$ is h-good, $f(a) = h^{-1}(a)$. As $g(a')$ is not h-good, $f(a') = g(a')$.

Therefore, $h^{-1}(a) = g(a')$. Call this element b^*.

As $g(a') = b^*$, $b^* \notin B_1$. But as $g(a)$ is h-good. Therefore,

$(h \odot g)^{-i}(b^*) \in B_1$ for some $i \in \mathbb{N}$.

Schröder-Bernstein

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Assuming $g(a')$ is not h-good, paths walked backwards from b^* lead to B_0.

But Assuming $g(a)$ is h-good, paths walked backwards from b^* lead to B_1.

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Contradiction! □

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HW □

Subset Take-away Game – David Gale

I am player 1 and you are player 2. We both have been given a set A. In each round, first I choose one subset of A and then you choose another subset of A. We stick to the following rules:

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(Source – Mathematics for Computer Science, 2012, by Eric Lehman and F Thomson Leighton)

CS 207 Discrete Mathematics – 2013-2014

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Mathematical Reasoning and Mathematical Objects

Lecture 6: Relations

Aug 06, 2013

Last few classes

Recap

- Proofs, proof methods.

Recap

- Proofs, proof methods.
- Sets and properties of sets

Recap

- Proofs, proof methods.
- Sets and properties of sets
- Functions, properties of functions

Recap

- Proofs, proof methods.
- Sets and properties of sets
- Functions, properties of functions
- Infinite sets and properties of infinite sets.

Today

- Relations: generalisations of functions

Today

- Relations: generalisations of functions
- Types and properties of relations

Today

- Relations: generalisations of functions
- Types and properties of relations
- Representation of functions - directed graphs

What are relations?

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- $R(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } a \leq b\}$. R is a relation on the set of integers under which aRb holds for two numbers $a, b \in \mathbb{Z}$ if and only if $a \leq b$.

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- Let S be a set $R(\mathcal{P}(S)) = \{(A, B) \mid A, B \in \mathcal{P}(S) \text{ and } A \subseteq B\}$.

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- Relational databases: practical examples of relations.

Properties of relations

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  - Is $R(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } a \leq b\}$ symmetric?
  - Is $R(\text{YourClass}) = \{(a, b) \mid a, b \in \text{YourClass and } a \text{ friend of } b\}$ symmetric?
  - Is $R(\mathcal{P}(S)) = \{(A, B) \mid A \cap B = \emptyset\}$ symmetric?

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  - Is $R(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } a \leq b\}$ transitive?
  - Is $R(\mathbb{N}) = \{(a, b) \mid a \ (mod \ b) \neq 0\}$ transitive?
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  Σ is a finite alphabet. Σ^* are strings of arbitrary length over Σ

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- $R(\mathbb{R}) = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } a^2 \leq b^2\}$

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CS 207 Discrete Mathematics – 2013-2014

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Mathematical Reasoning and Mathematical Objects

Lecture 7: Properties of equivalence relations and partial orders

August 08, 2013

Last time

Recap

- What are relaions?

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- What are different types of relations?

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- Representation of partial orders by graphs

Today

- Chains, anti-chains, properties of partial orders.

Today

- Chains, anti-chains, properties of partial orders.
- What are equivalence classes and properties of equivalence classes.

Posets, Chains and Anti-chians

Definition

A set S along with a relation $\preceq$, $(S, \preceq)$, is called a poset if $\preceq$ defines a partial order on S.

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Definition

Let $(S, \preceq)$ be a poset. A subset $A \subseteq S$ is called an anti-chain if no two elements of A are related to each other under $\preceq$.

Representing partial orders as directed graphs

What is a graph?

Representing partial orders as directed graphs

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Definition

A graph can be described by two sets: set V is called a set of vertices and set E is a subset of $V \times V$ and is called a set of edges, $G = (V, E)$.

Representing partial orders as directed graphs

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The graph is called directed if E is a set of ordered pairs.

What are the examples of graphs you may have seen:

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What are the examples of graphs you may have seen:

- Social network graphs
- Tum-tum route graphs
- ...

Representing partial orders as directed graphs

Let $S = \{1, 2, 3\}$. Recall the poset $(\mathcal{P}(S), \subseteq)$.

Representing partial orders as directed graphs

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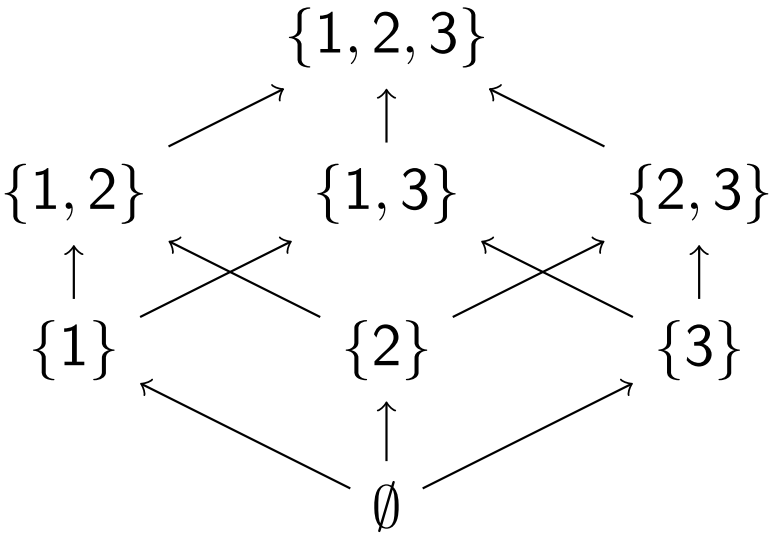

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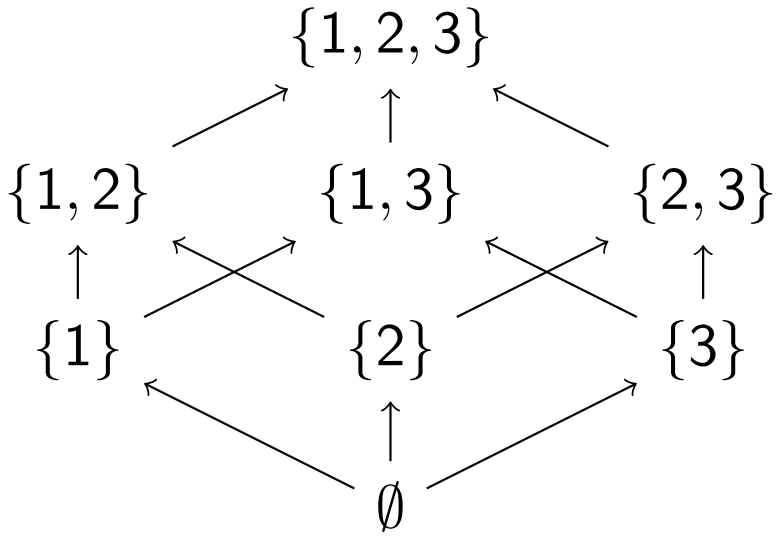

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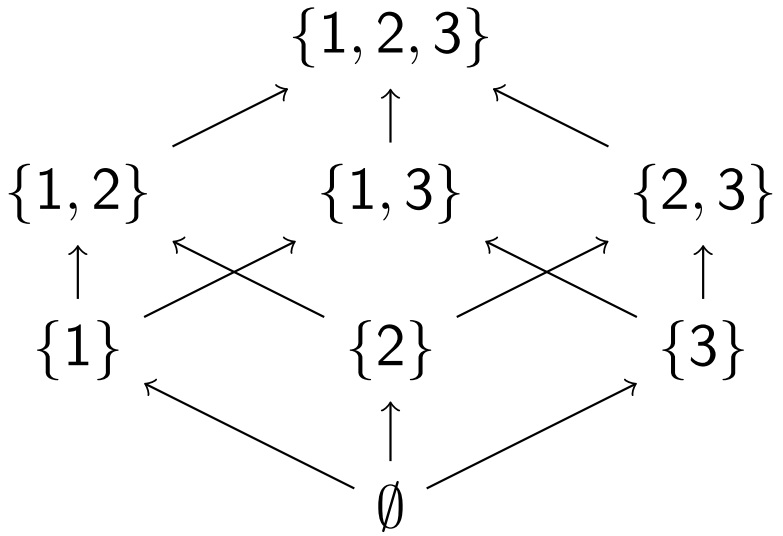

[CW] What are the chains in this poset?
[CW] What are the anti-chains in this poset?

Chains and anti-chains

Theorem

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If the largest chain in a poset $(S, \preceq)$ is of size m then S has at least m anti-chains. The size of an anti-chain is the number of elements in the chain.

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Theorem

If the largest chain in a poset $(S, \preceq)$ is of size m then S has at least m anti-chains.

Proof.

Let the chain be denoted as $a_1 \preceq a_2 \preceq \ldots \preceq a_m$. Now observe that every element of this chain, must go to different anti-chains.

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If the largest chain in a poset $(S, \preceq)$ is of size m then S has at least m anti-chains.

Proof.

Let the chain be denoted as $a_1 \preceq a_2 \preceq \ldots \preceq a_m$. Now observe that every element of this chain, must go to different anti-chains. Therefore, there are at least m anti-chains in $(S, \preceq)$. □

Chains and anti-chains

Theorem (Mirsky's theorem, 1971)

If the largest chain in a poset $(S, \preceq)$ is of size m then S can be partitioned into m anti-chains.

CS 207 Discrete Mathematics – 2013-2014

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Mathematical Reasoning and Mathematical Objects

Lecture 8: Properties of equivalence relations and partial orders

August 12, 2013

Last time

Recap

- What are graphs?

Recap

- What are graphs?
- Representation of partial orders by graphs.

Recap

- What are graphs?
- Representation of partial orders by graphs.
- Chains, anti-chains, properties of partial orders.

Recap

Definition

A set S along with a relation $\preceq$, $(S, \preceq)$, is called a poset if $\preceq$ defines a partial order on S.

Definition

If $(S, \preceq)$ is a poset and every pair of elements in S is comparable, then $(S, \preceq)$ is called a totally ordered set. A totally ordered set is called a *chain*.

Definition

Let $(S, \preceq)$ be a poset. A subset $A \subseteq S$ is called an anti-chain if no two (distinct) elements of A are related to each other under $\preceq$.

Recap

Theorem

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Today

- Proof of Mirsky's theorem

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- Proof of Mirsky's theorem
- What are equivalence classes and properties of equivalence classes.

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For each element $s \in S$, let C_s be the set of all chains that have s as the maximum element. And define $label(s) := max_{c \in C_s} \{\text{size}(c)\}$.

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[CW] For any $s \in S$, how large can $label(s)$ be?

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Let us now define sets $A_1, A_2, \ldots, A_m$ such that $A_i = \{x \mid label(x) = i\}$.

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□

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Properties of equivalence relations

Let R be an equivalence relation of A. Let elements of A be x, y, z etc.

Lemma

The following three are equivalent: *(a)* xRy, *(b)* $[x] = [y]$, *(c)* $[x] \cap [y] \neq \emptyset$.

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(a) $\Rightarrow$ (b): Say $z \in [x]$. But xRy. As xRy and R is symmetric, yRx. Therefore, yRx, xRz. R is transitive. Therefore, yRz, i.e. $z \in [y]$. This proves that $[x] \subseteq [y]$. The proof of $[y] \subseteq [x]$ is similar.

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(c) $\Rightarrow$ (a): Let $z \in [x] \cap [y]$. Therefore, xRz and yRz. But as R is symmetric, zRy. But R is also transitive. Therefore xRz and zRy imply xRy. □

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Sets $X_1, X_2, \ldots, X_m$ are said to partition a set X if

- $\forall i, j \in \{1, 2, \ldots, m\}, i \neq j : X_i \cap X_j = \emptyset$
- $\forall x \in X, \exists i \in \{1, 2, \ldots, m\} : x \in X_i$

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Let $[x] \neq [y]$ be two distinct equivalence classes of R. From the previous lemma $[x] \cap [y] = \emptyset$.

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R_A relates (x, y) if they belong to the same part in the partition of A.

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Let $R_A = \{(x, y) \mid \exists i : x, y \in A_i\}$.
R_A is reflexive. If $(x, y) \in R_A$ then even $(y, x) \in R_A$. Finally, if $(x, y) \in R_A$ then $\exists i : x, y \in A_i$. Let that index be called i_0. Now if $(y, z) \in R_A$ then both y, z must be in the same part of the partition. But we know that $y \in A_{i_0}$. Therefore, $z \in A_{i_0}$. Hence, $x, z \in A_{i_0}$ and hence $(x, z) \in R_A$. This proves that R_A is also transitive. □