CS 207 Discrete Mathematics – 2013-2014

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 10: Double counting

August 19, 2013

Course Outline

- Mathematical reasoning and mathematical objects
- Combinatorics

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  - What is a proof? Types of proof methods
  - Induction
  - Sets, relations, functions, partial orders, graphs
- Combinatorics
- Elements of graph theory
- Elements of abstract algebra

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Text: *Discrete Mathematics and its applictions, by Kenneth Rosen*
Chapter 2 : 2.1, 2.2, 2.3, Chapter 8 : 8.1, 8.5, 8.6
Class notes: uploaded on Moodle

- Combinatorics
- Elements of graph theory
- Elements of abstract algebra

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- Mathematical reasoning and mathematical objects
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  - Approximating sums and products
  - Pigeonhole principle
  - Recurrence relations and generating functions
  - Inclusion-exclusion principle
  - Elements of discrete probability
- Elements of graph theory
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    - The rest of pairs, $n^2 - n$ of them, may or may not be put.
    - Therefore, there are 2^{n^2-n} different reflexive relations.

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$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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  - As LHS and RHS are counting the same quantity they must be equal.

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$$S = 1 + \qquad x + \ldots + \qquad x^n \tag{1}$$

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  - $$\begin{aligned} S &= 1 + \quad x + \ldots + \quad x^n \qquad (1) \\ xS &= x + \quad x^2 + \ldots + \quad x^{n+1} \qquad (2) \end{aligned}$$

    (1)-(2) gives us: $(1-x)S = 1 - x^{n+1}$
  - Therefore, $S = \frac{1-x^{n+1}}{1-x}$.

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  - Ramsey proved that a draw is impossible!

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On various occasions different quantities may become interesting. Some may be easy to count directly. Some may require more thought.

[CW] Count the number of arrangements of wrongly addresses letters for $n = 4$.

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In this module, we will build some technqiues that will help in counting some quantities which are hard to count.

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Often to count a certain object, we will count some totally different object!

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Proof.

Recall $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

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Note that, LHS equals this quantity.

For the RHS, note that

- Either choose the mango in the basket and select $k-1$ apples from n apples in $\binom{n}{k-1}$ ways.
- Or leave out the mango from the basket and select k apples from n apples in $\binom{n}{k}$ ways.

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$\text{Degree}(v) := |\{u \mid (u, v) \in E\}|$

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- On the one hand this number is $\sum_{i=1}^{n} m_i$.

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- On the one hand this number is $\sum_{i=1}^{n} m_i$.
- On the other hand each handshake gives rise to two edges. So if X is the number of handshakes, then the number of edges is $2X$.

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Take back message: Counting the same quantity the number of directed edges in two different ways can be helpful!

Counting the same quantity in different ways

Lemma

Cosnider a class of m students. Every day after class 3 students stay back to clean the classes. At the end of the course, they realise that each pair of students stayed back exactly once. For how many days did the course run?

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Say the course ran for n days.

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Let P be the total number of distinct pairs of students.

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On the other hand, each day 3 pairs of students stay back together. As there are n days, $P = 3n$.

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On the other hand, each day 3 pairs of students stay back together. As there are n days, $P = 3n$.
$\therefore n = \frac{m(m-1)}{6}$. $\square$

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Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 11: Counting the same object in two ways
August 20, 2013

Last time

Recap

- Counting the same object in two different ways
  - Basic counting
  - $k\binom{n}{k} = n\binom{n-1}{k-1}$
  - $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
  - The number of people who shake hands odd number of times is even.

Today

- Counting the number of labelled trees – Cayley's number.

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- Counting the number of labelled trees – Cayley's number.
- How large/small is $n!$? – approximating $n!$ [Stirling's approximation]

Counting labeled trees – Cayley's number

Recall

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- What are directed and undirected graphs?
- What is a cycle in a graph?
- What is a tree?

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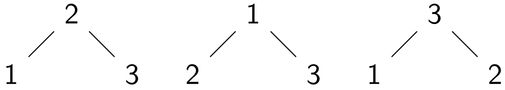
What is a labeled tree?

Example: Labeled trees on 3 vertices



How many labeled trees on n vertices?

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Theorem (Cayley)

There are n^{n-2} labeled tree on n vertices.

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Count one quantity in order to count the other

CS 207 Discrete Mathematics – 2013-2014

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 12: Generating functions

August 22, 2013

Last time

Recap

- Cayley's number: the number of labelled trees equals n^{n-2}.

Today

- Recurrences and generating functions.

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Solving recurrences

- Let $F(n)$ denote the nth Fibonacci number. Compute $F(n)$.

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In general, let $C(n)$ be the number of ways of doing this.

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How to solve this recurrence? Using generating functions, of course!

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Theorem (Extended Binomial Theorem)

Let $\alpha \in \mathbb{R}^+$. $(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, *where* $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}$

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- The coefficient of t^n is
  $C(n) = -\frac{1}{2}\binom{1/2}{n}(-4)^n = -\frac{1}{2}\left(\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-n+1)\right)\frac{(-4)^n}{n!}$

Catalan number

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Catalan Number

Theorem (n-th Catalan Number)

If the recurrence for $C(n)$ is given as follows:

$$C(n) = \sum_{i=1}^{n-1} C(i)C(n-i) \qquad \textit{for } n > 1$$

then

$$C(n) = \frac{1}{n}\binom{2n-2}{n-1}$$

CS 207 Discrete Mathematics – 2013-2014

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 13: Catalan numbers, derrangements
August 26, 2013

Last time

Recap

- Introduction to recurrences and generating functions
- Compute the n-th Catalan number using generating functions

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Today

- Coming up with recurrence relations.
- Computing the number of derrangements
- Exponential generating functions

Find recurrence relations

- [CW] What is the number of different ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines?

Find recurrence relations

- [CW] What is the number of different ways a convex polygon with $n + 2$ sides can be cut into triangles by connecting vertices with straight lines?
- [CW] What is the number of monotonic paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal?

Derrangements

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CS 207 Discrete Mathematics – 2013-2014

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nutan@cse.iitb.ac.in

Combinatorics

Lecture 14: Derrangements and estimating $n!$

August 27, 2013

Last time

Recap

- Coming up with recurrences

Today

- Computing the number of derrangements
- Exponential generating functions
- Estimating $n!$

Closed form for $D(n)$

Theorem

Let $D(n)$ denote the number of derrangements for n elements then

$$D(n) = n! \left(\sum_{i=0}^{n} \frac{(-1)^i}{i!} \right)$$

Proof.

We will prove that RHS has the same recurrence as LHS and RHS matches with LHS for $n = 0, 1$.

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$$= (n-1)\left[(n-1)!\left(\sum_{i=0}^{n-1} \frac{(-1)^i}{i!}\right) + (n-2)!\left(\sum_{i=0}^{n-2} \frac{(-1)^i}{i!}\right)\right]$$

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after come calculations

$= f(n)$

$\square$

Computing $D(n)$ using generating functions

Exponential generating functions(EGF) – For arrangements, need a $n!$ normaliser for the recurrence to work out.

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Let $D(n)$ be the n coefficient of the exponential generating functions.

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Any permutation of $[n]$ can be obtained by

- first picking a subset $S \subseteq [n]$.
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Every permutation is generated in this manner.

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- $P(t) = \phi(t) \cdot I(t) \therefore \phi(t) = P(t)/I(t) = \frac{e^{-t}}{1-t}$

$$\phi(t) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} t^n \right)$$

- Now, $D(n) =$ the coefficient of $(t^n/n!)$ is:

$$= \left[(-1)^n + (-1)^{n-1} \cdot n + (-1)^{n-2} \cdot n \cdot n - 1 + \ldots + \right.$$
$$\left. (-1) \cdot n \cdot n - 1 \cdot \ldots \cdot 2 + n! \right]$$
$$= \left(n! \sum_{i=1}^{n} \frac{(-1)^i}{i!} \right)$$

How large is $D(n)$ as compared to $n!$

Theorem

$|D(n) - \frac{n!}{e}| \leq \frac{1}{2} \quad \forall n \geq 1$

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$$\leq \left|\frac{(-1)^{n+1}}{(n+1)!}\right|$$

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$$\leq \left| \frac{(-1)^{n+1}}{(n+1)!} \right| \quad [CW] why?$$

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\begin{aligned}
\left| D(n) - \frac{n!}{e} \right| &= n! \left| \left(\sum_{i=0}^{n} \frac{(-1)^i}{i!} - \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \right) \right| \\
&\leq \left| \frac{(-1)^{n+1}}{(n+1)!} \right| \\
&= \frac{1}{n+1}
\end{aligned}
$$

How large is $D(n)$ as compared to $n!$

Theorem

$|D(n) - \frac{n!}{e}| \leq \frac{1}{2} \quad \forall n \geq 1$

Proof.

We consider the Taylor series expansion of $1/e = \sum_{i=0}^{\infty} (-1)^i / i!$

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&\leq \frac{1}{2} \quad n > 1
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&\leq \frac{1}{2} \quad n > 1
\end{aligned}$$

CS 207 Discrete Mathematics – 2013-2014

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 15: Estimating $n!$

August 29, 2013

Last time

Recap

- Coming up with recurrences
- Computing the number of derrangements
- Exponential generating functions

Today

- Estimating $n!$

Estimating $n!$

- How large/small is $n!$? – approximating $n!$ [Stirling's approximation]

Estimating $n!$

- How large/small is $n!$? – approximating $n!$ [Stirling's approximation]
- Counting the number of labelled trees – Cayley's number.

Estimating $n!$

- Easy to see that $n! \leq n^n$

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Estimating $n!$

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- However, is this tight? Of course not!
  Can we quantify how much more is n^n as compared to $n!$?
  Can we bound $n!$ by a quantity, say Q, so that for some small enough α, $\alpha Q \leq n! \leq Q$?

Theorem (Stirling's approximation)

$e(n/e)^n \leq n! \leq ne(n/e)^n$, i.e. $Q = e(n/e)^n$, and $\alpha = 1/n$

Estimating $n!$

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Estimating $n!$

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Proof.

Let $S = \log(n!)$

Estimating $n!$

Theorem (Stirling's approximation)

$e(n/e)^n \leq n! \leq ne(n/e)^n$

Proof.

Let $S = \log(n!) = \sum_{i=1}^{n} \log i$.

Estimating $n!$

Theorem (Stirling's approximation)

$e(n/e)^n \leq n! \leq ne(n/e)^n$

Proof.

Let $S = \log(n!) = \sum_{i=1}^{n} \log i$. We will bound S using the natural log. From the figure on the board:

$$\sum_{i=1}^{n-1} \log i \leq \int_1^n \log x \; dx$$

$$\begin{aligned} S &\leq \int_1^n \log x \; dx + \log n \\ &= (x \log x - x)|_1^n + \log n \\ &= n \log n - n + 1 + \log n \end{aligned}$$

Estimating $n!$

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Proof.

Let $S = \log(n!) = \sum_{i=1}^{n} \log i$. We will bound S using the natural log.
$\therefore S \leq n \log n - n + 1 + \log n$
raising both sides to the power of e, we get

$$\begin{aligned} n! &\leq e^{(n+1)\log n - (n-1)} \\ &= n^{n+1}/e^{n-1} \\ &= ne(n/e)^n \end{aligned}$$

$\square$

Estimating $n!$

Theorem (Stirling's approximation)

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Let $S = \log(n!) = \sum_{i=1}^{n} \log i$. We will bound S using the natural log. From the figure on the board:

$$\sum_{i=1}^{n} \log i \geq \int_1^n \log x \; dx$$

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$$\begin{aligned} n! &\geq e^{n \log n - (n-1)} \\ &= e(n/e)^n \end{aligned}$$

$\square$

CS 207 Discrete Mathematics – 2013-2014

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 16: Inclusion and exclusion

September 02, 2013

Last time

Recap

Last few lectures (module 2):

- Double counting
- Coming up with recurrences and solving recurrences (generating functions)
- Exponential generating functions
- Estimating n!

Today

- The principle of inclusion and exclusion (PIE).
- Computing the number of surjections using PIE.
- Proof of PIE.

The principle of inclusion and exclusion (PIE)

Theorem (Principle of inclusion exclusion)

Let $A_1, A_2, \ldots, A_n$ be the finite sets from a universe U.

$$|\cup_{i=1}^{n} A_i| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} |\cap_{i \in I} A_i|$$

The principle of inclusion and exclusion (PIE)

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$$|\cup_{i=1}^{n} A_i| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \binom{[n]}{k}} |\cap_{i \in I} A_i|$$

Number of surjections

How many surjections from $[n]$ to $[k]$?

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How many functions from $[n]$ to $[k]$?

Number of surjections

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How many functions from $[n]$ to $[k]$? k^n

Number of surjections

How many surjections from $[n]$ to $[k]$?
How many functions from $[n]$ to $[k]$?
Let

$$A_i = \{f : [n] \to [k] \mid \text{Range}(f) \subseteq [k] \setminus \{i\}\}$$

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Observe that $\forall I \subseteq [k]$, $|\cap_{i\in I} A_i| = (k - |I|)^n$

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Proof of PIE

Theorem (Principle of inclusion exclusion)

Let $A_1, A_2, \ldots, A_n$ be the finite sets from a universe U.

$$|\cup_{i=1}^{n} A_i| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} |\cap_{i \in I} A_i|$$

$$|\cup_{i=1}^{n} A_i| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \binom{[n]}{k}} |\cap_{i \in I} A_i|$$

We give the proof by induction on n.
For $n = 2$, the theorem says $|A \cup B| = |A| + |B| - |A \cap B|$.
Let us assume that the theorem holds for $n - 1$.
Let $A = \cup_{i=1}^{n-1} A_i$ and let $B = A_n$.

cotn.

Proof of PIE

$$|\cup_{i=1}^{n} A_i| \quad = \qquad\qquad |A \cup B|$$

Proof of PIE

$$|\cup_{i=1}^{n} A_i| = |A \cup B|$$

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Proof of PIE

$$
\begin{aligned}
|\cup_{i=1}^{n} A_i| &= |A| + |B| - |A \cap B| \\
|\cup_{i=1}^{n} A_i| &= |\cup_{i=1}^{n-1} A_i| + |A_n| - |\cup_{i=1}^{n-1} A_i \cap A_n|
\end{aligned}
$$

Proof of PIE

$$|\cup_{i=1}^{n} A_i| = |\cup_{i=1}^{n-1} A_i| + |A_n| - |\cup_{i=1}^{n-1} A_i \cap A_n|$$

$$|\cup_{i=1}^{n} A_i| = |\cup_{i=1}^{n-1} A_i| + |A_n| - |\cup_{i=1}^{n-1} (A_i \cap A_n)|$$

Proof of PIE

$$|\cup_{i=1}^{n} A_i| = |\cup_{i=1}^{n-1} A_i| + |A_n| - |\cup_{i=1}^{n-1} (A_i \cap A_n)|$$

By IH

Proof of PIE

$$|\cup_{i=1}^{n} A_i| = |\cup_{i=1}^{n-1} A_i| + |A_n| - |\cup_{i=1}^{n-1} (A_i \cap A_n)|$$

By IH

$$\begin{aligned}|\cup_{i=1}^{n} A_i| = {} & \textstyle\sum_{k=1}^{n-1}(-1)^{k-1} \sum_{I\in\binom{[n-1]}{k}} |\cap_{i\in I} A_i| \\ & + |A_n| - \textstyle\sum_{k=1}^{n-1}(-1)^{k-1} \sum_{I\in\binom{[n-1]}{k}} |\cap_{i\in I\cup\{n\}} A_i|\end{aligned}$$

Proof of PIE

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|\cup_{i=1}^{n} A_i| &= \sum_{k=1}^{n-1}(-1)^{k-1}\sum_{I\in\binom{[n-1]}{k}} |\cap_{i\in I} A_i| \\
&\quad +|A_n| - \sum_{k=1}^{n-1}(-1)^{k-1}\sum_{I\in\binom{[n-1]}{k}} |\cap_{i\in I\cup\{n\}} A_i|
\end{aligned}$$

$$\begin{aligned}
|\cup_{i=1}^{n} A_i| &= \left(\sum_{I\in\binom{[n-1]}{1}} |\cap_{i\in I} A_i| + |A_n|\right) \\
&\quad +\sum_{k=2}^{n-1}\left[(-1)^{k-1}\sum_{I\in\binom{[n-1]}{k}} |\cap_{i\in I} A_i| - (-1)^{k-2}\sum_{I\in\binom{[n-1]}{k-1}} |\cap_{i\in I\cup\{n\}} A_i|\right]
\end{aligned}$$

Proof of PIE

$$
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|\cup_{i=1}^{n} A_i| &= (\textstyle\sum_{I\in\binom{[n]}{1}} |\cap_{i\in I} A_i| \\
&+ \textstyle\sum_{k=2}^{n-1]} \left[(-1)^{k-1} \sum_{I\in\binom{[n-1}{k}} |\cap_{i\in I} A_i| + (-1)^{k-1} \sum_{I\in\binom{[n-1]}{k-1}} |\cap_{i\in I\cup\{n\}} A_i|\right]
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\end{aligned}
$$

Proof of PIE

$$
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|\cup_{i=1}^{n} A_i| &= \left(\sum_{I \in \binom{[n-1]}{1}} |\cap_{i\in I} A_i| + |A_n|\right) \\
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|\cup_{i=1}^{n} A_i| &= \left(\sum_{I \in \binom{[n]}{1}} |\cap_{i\in I} A_i|\right. \\
&\quad + \sum_{k=2}^{n-1]} \left[(-1)^{k-1} \sum_{I \in \binom{[n-1]}{k}} |\cap_{i\in I} A_i| + (-1)^{k-1} \sum_{I \in \binom{[n-1]}{k-1}} |\cap_{i\in I\cup\{n\}} A_i|\right] \\
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|\cup_{i=1}^{n} A_i| &= \sum_{k=1}^{n} \left[(-1)^{k-1} \sum_{I \in \binom{[n]}{k}} |\cap_{i\in I} A_i|\right]
\end{aligned}
$$

CS 207 Discrete Mathematics – 2013-2014

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 17: Bell numbers and Stirling's number

September 03, 2013

Last time

Recap

- The principle of inclusion and exclusion (PIE).
- Computing the number of surjections using PIE.
- Proof of PIE.

Today

- Derangements
- Counting the number of partitions of a set
- Stirling's numbers.

Number of derangements

Recall that the number of derangements of n letters, denoted as $D(n)$ is given by:

$$n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

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Let A_i denote the set of permutations that map i to i.

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$$D(n) \quad = n! - |\cup_{i \in [n]} A_i|$$

Number of derangements

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Let A_i denote the set of permutations that map i to i.
Observe that $\forall I \subseteq [n]\ |\cap_{i \in I} A_i| = (n - |I|)!$

$$\begin{aligned} D(n) &= n! - |\cup_{i\in[n]} A_i| \\ &= n! - \textstyle\sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} (n - |I|)! \end{aligned}$$

Number of derangements

Recall that the number of derangements of n letters, denoted as $D(n)$ is given by:

$$n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

Let A_i denote the set of permutations that map i to i.
Observe that $\forall I \subseteq [n] \; |\cap_{i \in I} A_i| = (n - |I|))!$

$$\begin{aligned} D(n) &= n! - |\cup_{i \in [n]} A_i| \\ &= n! - \textstyle\sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} (n - |I|)! \\ &= \textstyle\sum_{I \subseteq [n]} (-1)^| I|(n - |I|)! \end{aligned}$$

Number of derangements

Recall that the number of derangements of n letters, denoted as $D(n)$ is given by:

$$n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

Let A_i denote the set of permutations that map i to i.
Observe that $\forall I \subseteq [n] \mid \cap_{i \in I} A_i| = (n - |I|))!$

$$
\begin{aligned}
D(n) &= n! - |\cup_{i\in[n]} A_i| \\
&= n! - \textstyle\sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} (n - |I|)! \\
&= \textstyle\sum_{I \subseteq [n]} (-1)^{|} I|(n - |I|)! \\
&= \textstyle\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!
\end{aligned}
$$

Number of derangements

Recall that the number of derangements of n letters, denoted as $D(n)$ is given by:

$$n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

Let A_i denote the set of permutations that map i to i.
Observe that $\forall I \subseteq [n] \; | \cap_{i \in I} A_i| = (n - |I|))!$

$$\begin{aligned}
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&= n! - \textstyle\sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|-1} (n - |I|)! \\
&= \textstyle\sum_{I \subseteq [n]} (-1)^| I|(n - |I|)! \\
&= \textstyle\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! \\
&= n! \textstyle\sum_{i=0}^{n} \frac{(-1)^i}{i!}
\end{aligned}$$

Bell numbers

Let $B(n)$ be #partitions of a set of size n such that each part is non-empty.

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Example: $n = 3$
$\{\{1\}, \{2\}, \{3\}\}$ $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$ $\{\{1, 2, 3\}\}$
That is, $B(3) = 5$.

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What is the recurrence for $B(n)$?

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$$\forall n \geq 1 \quad B(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k)$$

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Also $X' \subseteq \{1, 2, \ldots, n-1\}$.

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In any partition, a unique part contains the element n.
Let $X = X' \cup \{n\}$. Now $|X'| = k - 1$ if $|X| = k$.
Also $X' \subseteq \{1, 2, \ldots, n-1\}$.
The rest of the $n - k$ elements can be partitioned into $B(n-k)$ ways.

Stirling's number

Let $S(n, k)$ be #partitions of a set of size n into k parts such that each part is non-empty.

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Let $f : [n] \to [k]$ be a surjection.

It defines a partition of n elements into k parts:
$\{f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k)\}$

For each partition, any possible ordering of parts gives rise to a surjection. And the number of ways of ordering the parts is $k!$.

CS 207 Discrete Mathematics – 2013-2014

Nutan Limaye

Indian Institute of Technology, Bombay
nutan@cse.iitb.ac.in

Combinatorics

Lecture 18: Pigeon Hole Principle (PHP)

September 17, 2013

Last time

Recap

Last few classes (Module 2)

- Counting in two ways
- Recurrences and generating functions
- The principle of inclusion and exclusion (PIE).

Today

- Pigeon hole principle (PHP)
- Applications of PHP

Pigeon hole principle

Theorem (PHP)

Let $n, k \in mathbbN$ and let there be n objects and k bags then there exists a bag with at least $\lfloor \frac{n}{k} \rfloor$ objects.

Pigeon hole principle

Theorem (PHP)

Let $n, k \in mathbbN$ and let there be n objects and k bags then there exists a bag with at least $\lfloor \frac{n}{k} \rfloor$ objects.

Suppose not. Then each bag has strictly less than $\lfloor \frac{n}{k} \rfloor$ objects. Therefore, totally there can be strictly less than n objects, which is a contradiction.

Application 1

Game: Give me 10 numbers such that no subsequence of length 4 which is increasing or decreasing.

Application 1

Lemma

Suppose there are $n^2 + 1$ numbers (all distinct) then there exists either an increasing or decreasing subsequence of length $n + 1$.

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Suppose there are $n^2 + 1$ numbers (all distinct) then there exists either an increasing or decreasing subsequence of length $n + 1$.

Let $a_1, a_2, \ldots, a_{n^2+1}$ be any sequence with distinct numbers.

Application 1

Lemma

Suppose there are $n^2 + 1$ numbers (all distinct) then there exists either an increasing or decreasing subsequence of length $n + 1$.

Let $a_1, a_2, \ldots, a_{n^2+1}$ be any sequence with distinct numbers.
Let for each $j \in [n^2 + 1]$,

I_j := length of the longest increasing subsequence starting at a_j
D_j := length of the longest decreasing subsequence starting at a_j

Application 1

Lemma

Suppose there are n^2+1 numbers (all distinct) then there exists either an increasing or decreasing subsequence of length $n+1$.

Let $a_1, a_2, \ldots, a_{n^2+1}$ be any sequence with distinct numbers.
Let for each $j \in [n^2+1]$,

I_j := length of the longest increasing subsequence starting at a_j

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Note that for each $j \in [n^2+1]$, $I_j \leq n$ and $D_j \leq n$.

Application 1

Lemma

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Note that for each $j \in [n^2+1]$, $I_j \leq n$ and $D_j \leq n$.
Therefore, by PHP, there exists $k \neq l$ such that $(I_k, D_k) = (I_l, D_l)$).
Suppose $k < l$.

Application 1

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Let $a_1, a_2, \ldots, a_{n^2+1}$ be any sequence with distinct numbers.
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Suppose $k < l$.

Case 1 ($a_k > a_l$): then $D_k = D_l + 1$, i.e. contradiction

Application 1

Lemma

Suppose there are $n^2 + 1$ numbers (all distinct) then there exists either an increasing or decreasing subsequence of length $n + 1$.

Let $a_1, a_2, \ldots, a_{n^2+1}$ be any sequence with distinct numbers.
Let for each $j \in [n^2 + 1]$,

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Note that for each $j \in [n^2 + 1]$, $I_j \leq n$ and $D_j \leq n$.
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Case 1 ($a_k > a_l$): then $D_k = D_l + 1$, i.e. contradiction
Case 1 ($a_k < a_l$): then $I_k = I_l + 1$, i.e. contradiction

A version of PHP

Theorem

Suppose there are $n \geq 1 + r(l - 1)$ *ojects which are colored with* r *different colors. Then there exist* l *objects all with the same color.*

A version of PHP

Theorem

Suppose there are $n \geq 1 + r(l - 1)$ ojects which are colored with r different colors. Then there exist l objects all with the same color.

Proof [*HW*]

Recall the coloring game

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Lemma

Any 2-coloring of edges of a graph on 6 vertices has a monochromatic traingle.

Recall the coloring game

Lemma

Any 2-coloring of edges of a graph on 6 vertices has a monochromatic traingle.

Proof [*HW*]

Another version of PHP

Theorem

Suppose $n \geq k_1 + k_2 + \ldots + k_r - r + 1$, *and let* n *objects be put into* r *bags then for some* $i \in [r]$ *such that the* i*th bag has* k_i *objects in it.*

Another version of PHP

Theorem

Suppose $n \geq k_1 + k_2 + \ldots + k_r - r + 1$*, and let* n *objects be put into* r *bags then for some* $i \in [r]$ *such that the* i*th bag has* k_i *objects in it.*

Proof [*HW*]

Coloring the edges of a graph on 10 points

Lemma

Any 2-coloring of a graph on 10 vertices has either a monochromatic triangle or a monochromatic complete graph on 4 vertices.

Coloring the edges of a graph on 10 points

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Proof [*HW*]

CS 207

Discrete Structures

Nutan Limaye

19 SEP 2013

Last Class

1. PHP

2. Applications of PHP

    - inc/dec subsequences
    - 2-colorings of graphs.

Today :

More about 2-colorings of graph.

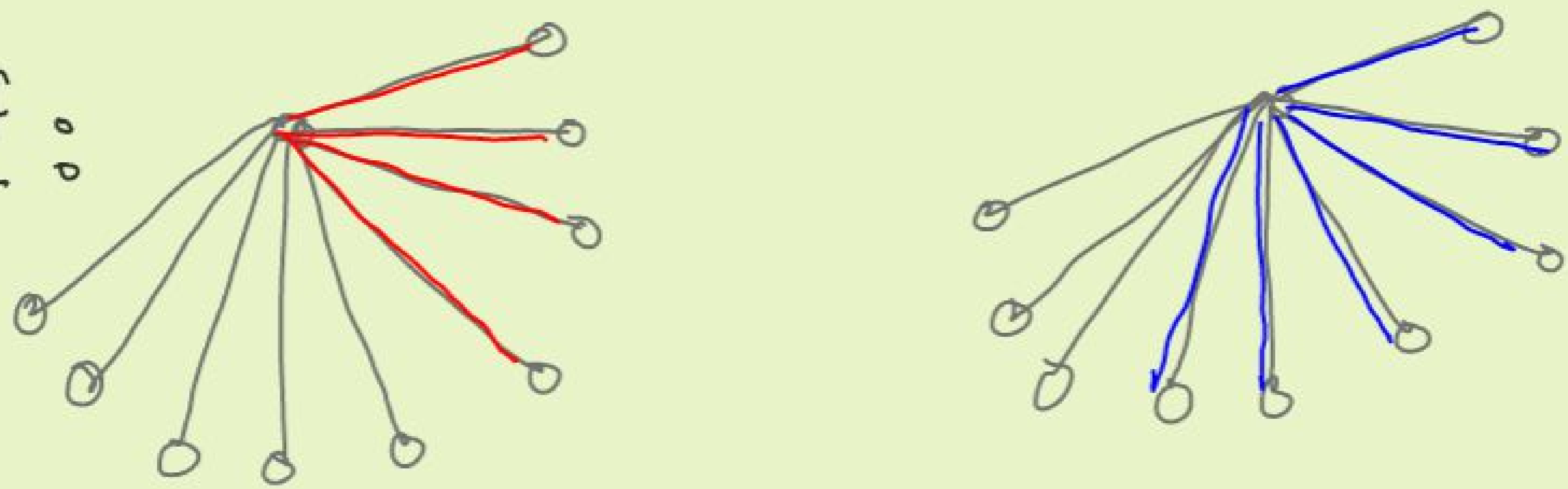
- Ramsey's number.

Any 2-coloring of the edges of a graph on 10 vertices has

- either a triangle of red color
- or a complete graph on 4 ver' with blue color.

Any 2-coloring of the edges of a graph on 10 vertices has
- either a triangle of red color
- or a complete graph on 4 ver' with blue color.

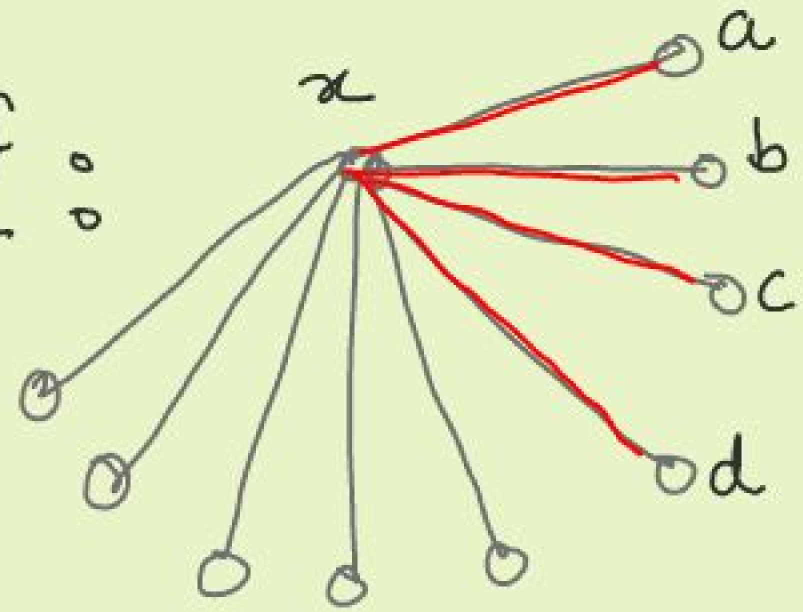
<u>proof</u> :



By PHP
- either 4 edges with red color
- or 6 edges with blue color

Any 2-coloring of the edges of a graph on 10 vertices has – either a triangle of red color
– or a complete graph on 4 ver' with blue color.

<u>proof</u> :

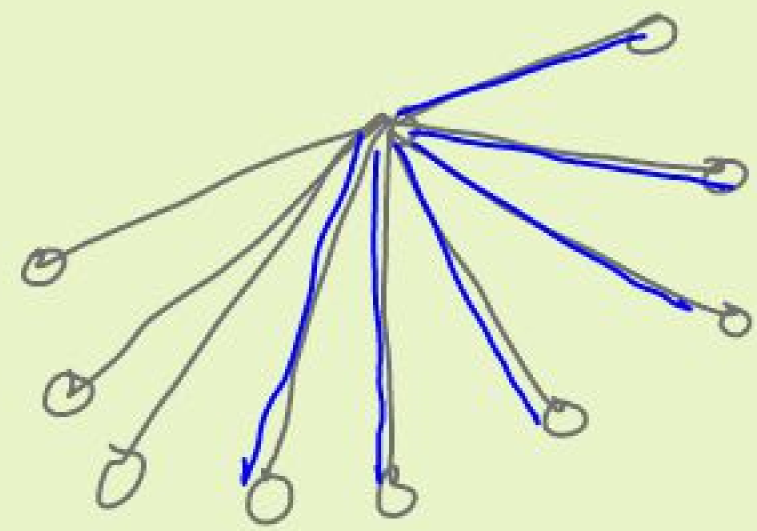


- either one of the edges btw a, b, c, d is red
- or all are blue.

∴ in this case done.

Any 2-coloring of the edges of a graph on 10 vertices has – either a triangle of red color
– or a complete graph on 4 ver' with blue color.

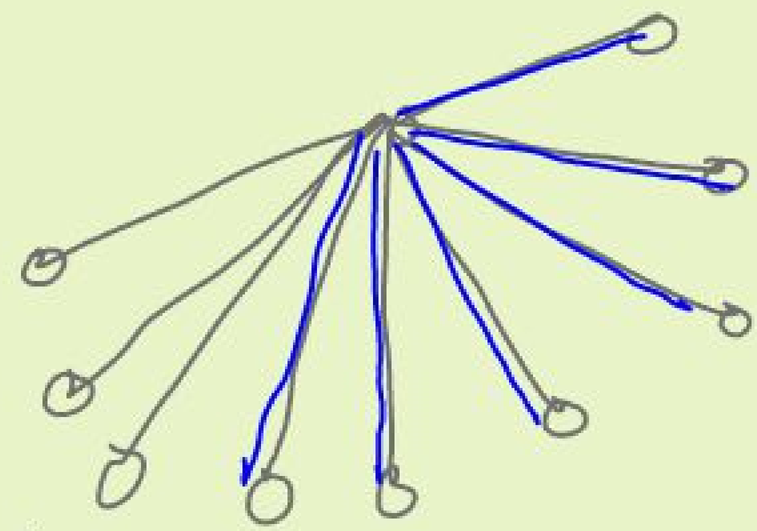
<u>proof</u> : ∘ apply the theorem from the prev class



Any 2-coloring of a graph on 6 vertices has a monochromatic triangle.

Any 2-coloring of the edges of a graph on 10 vertices has
- either a triangle of red color
- or a complete graph on 4 ver' with blue color.

proof :
- apply the theorem from the prev class



- ∴ either a red triangle
- or a blue triangle

∴ we are again done

□

Any 2-coloring of the edges of a graph on ~~10~~ 9 vertices has – either a triangle of red color

– or a complete graph on 4 ver' with blue color.

Any 2-coloring of the edges of a graph on 9 vertices has – either a triangle of red color

– or a complete graph on 4 ver' with blue color.

<u>proof</u>: The same proof as above will not go through if $\forall$ vertices x,

3 red edges

& 5 blue edges

Any 2-coloring of the edges of a graph on 9 vertices has – either a triangle of red color

– or a complete graph on 4 ver' with blue color.

<u>proof</u>: The same proof as above will not go through if $\forall$ vertices x,

3 red edges

& 5 blue edges

But now recall the hand shake lemma.

Handshake lemma:

In any graph the # vertices having odd degree is even.

Any 2-coloring of the edges of a graph on 9 vertices has
- either a triangle of red color
- or a complete graph on 4 ver' with blue color.

<u>proof</u>: The same proof as above will not go through if $\forall$ vertices x,

3 red edges

&

5 blue edges

By hand-shake lemma, this is not possible

$\square$

In general

How many vertices should a graph have so that so that any 2-coloring of its edges has

- k-sized complete graph of color red or
- l-sized complete graph of color blue.

Can we study this in general?

In general

How many vertices should a graph have so that so that any 2-coloring of its edges has

- k-sized complete graph of color red or
- l-sized complete graph of color blue.

Can we study this in general?

$R(k,l) \leftarrow$ Ramsey number.

Recall:

$$R(3,3) = 6$$

$$R(3,4) = 9$$

$$\vdots$$

$$\boxed{R(a_1, a_2) \leq \binom{a_1 + a_2 - 2}{a_1 - 1}}$$

Upper bound on Ramsey number.

Upper bound on Ramsey Number

Claim

$$\boxed{R(a_1, a_2) = \binom{a_1 + a_2 - 2}{a_1 - 1}}$$

CS 207

Discrete Structures

Nutan Limaye

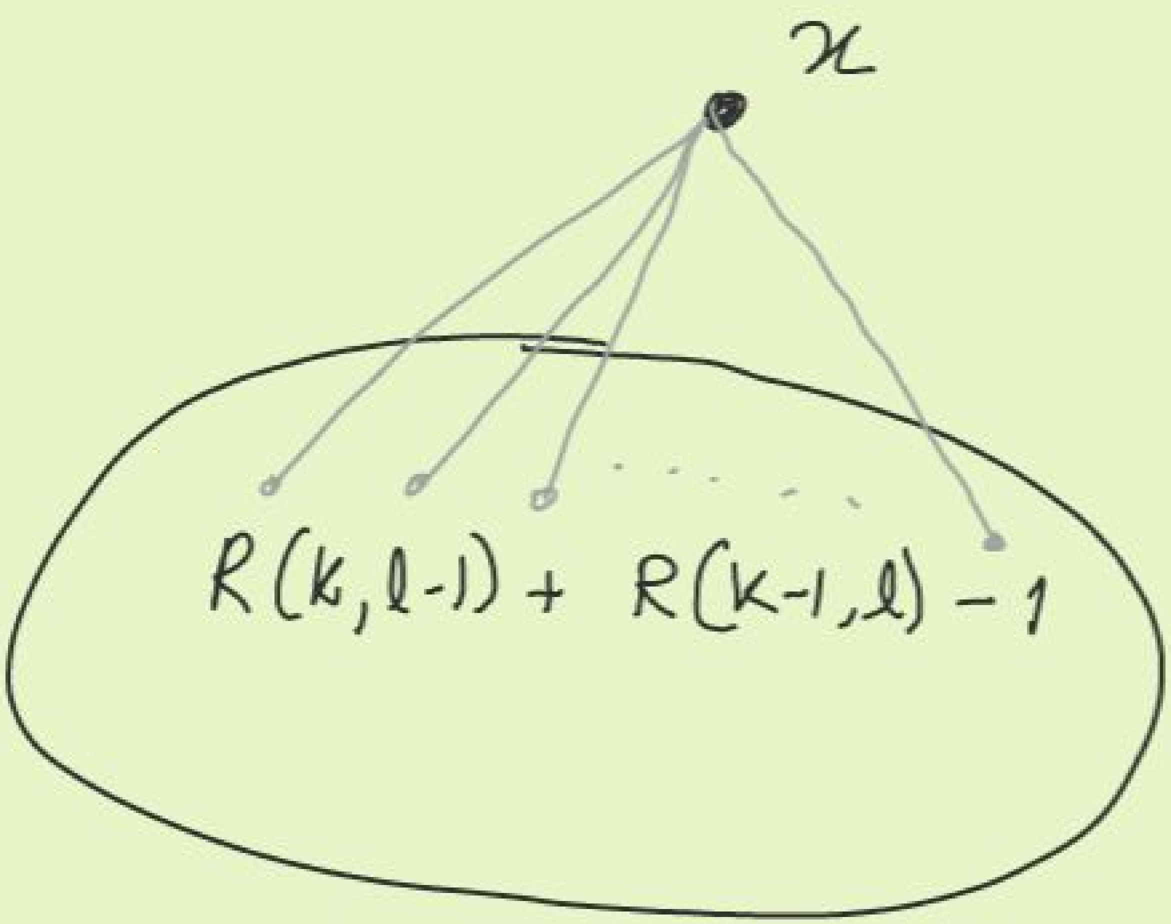
23 SEP 2013

Last Class

1. 2-colorings of edges of graphs (Ramsey Nos).

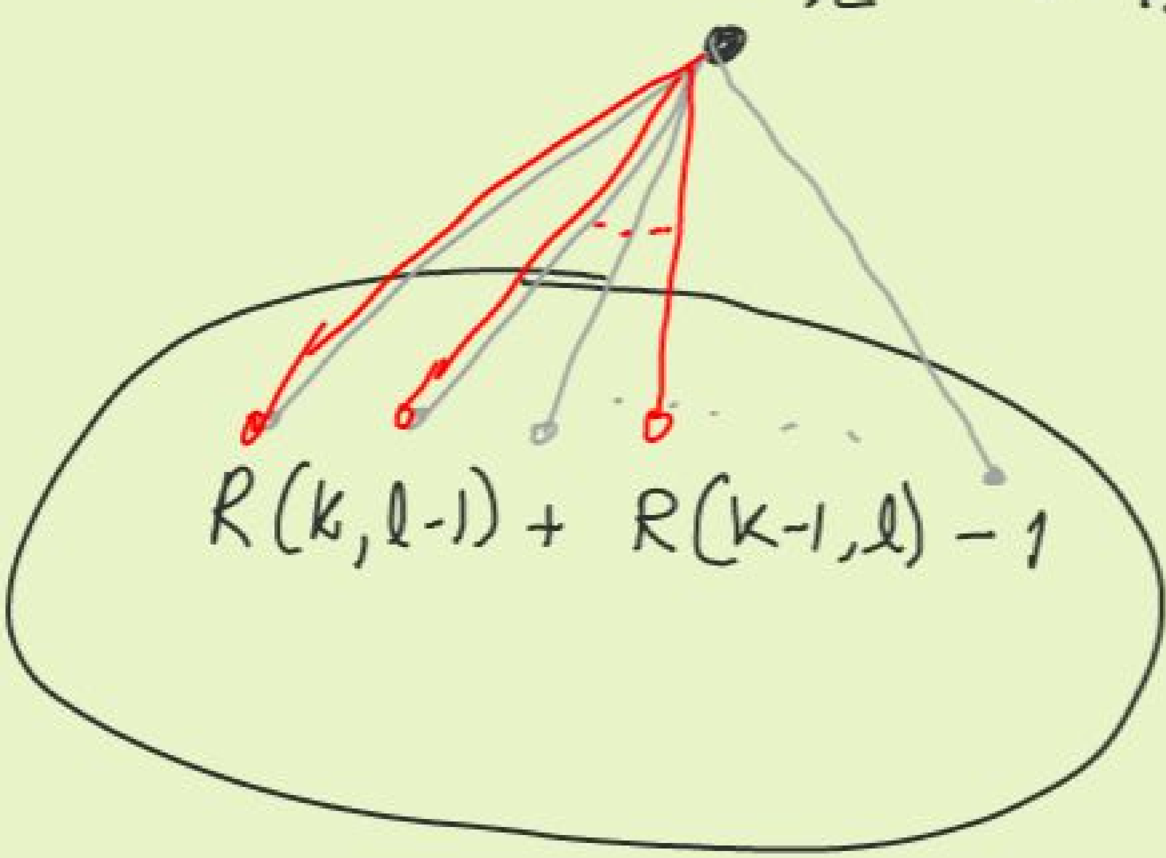
2. $R(k,l) \leq \binom{k+l-2}{k-1}$

Proof of $R(k,l) \leq \binom{k+l-2}{k-1}$



Proof of $R(k,l) \leq \binom{k+l-2}{k-1}$

x $\circ$ $\geq R(k-1,l)$ red edges



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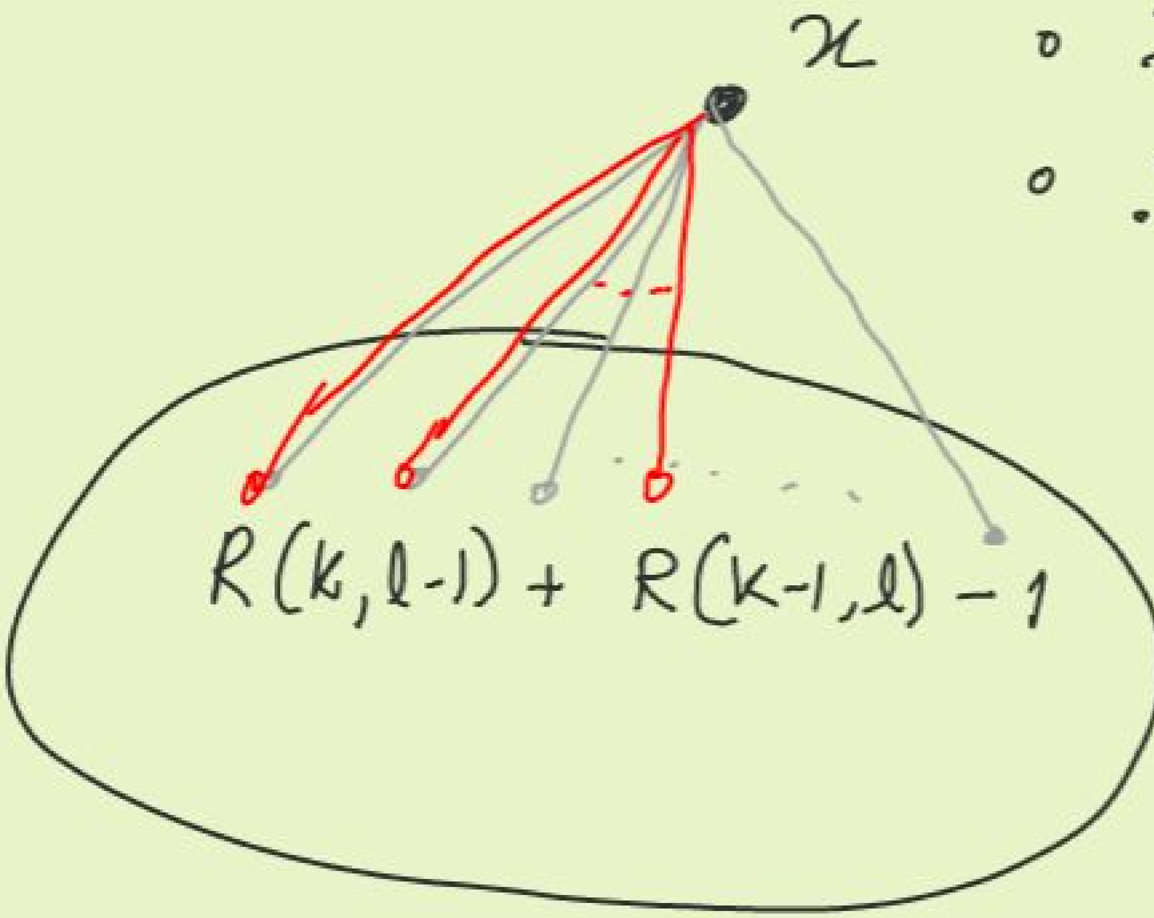
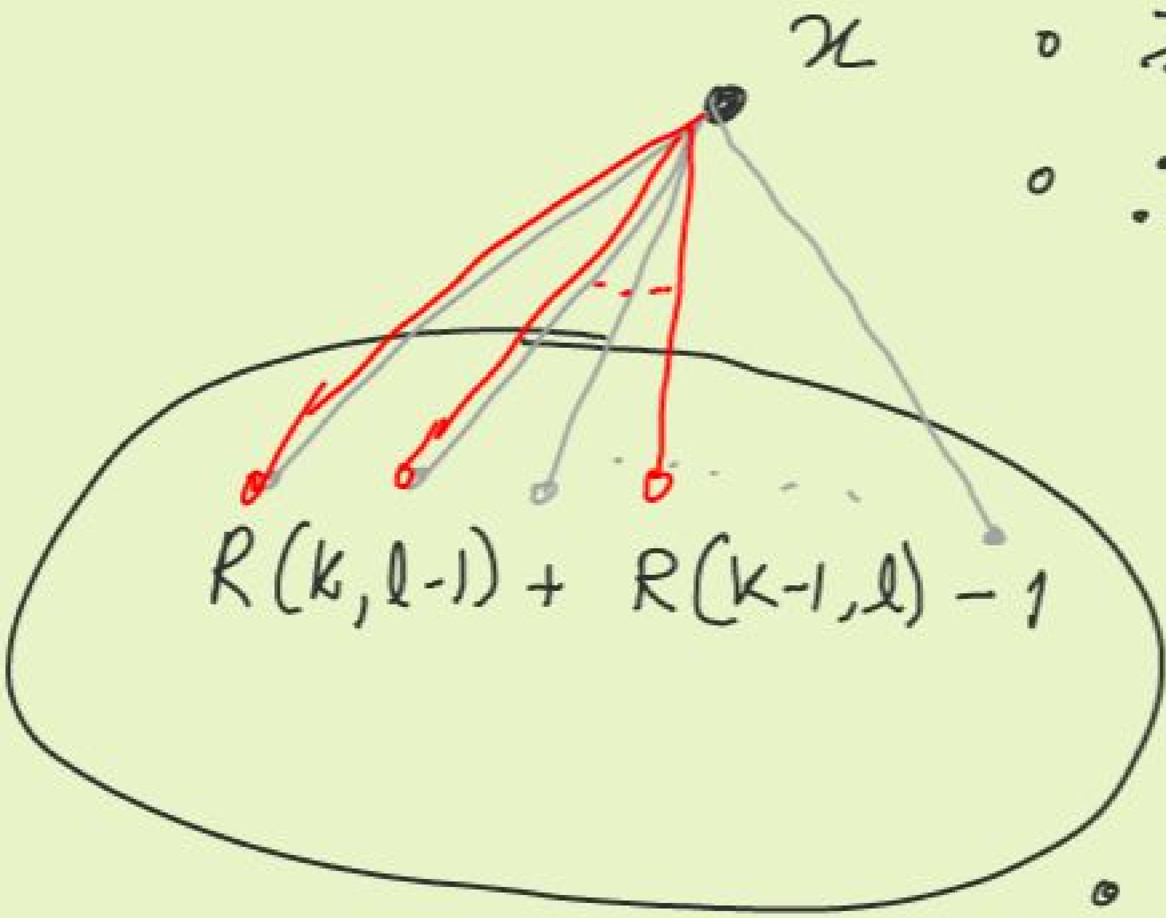


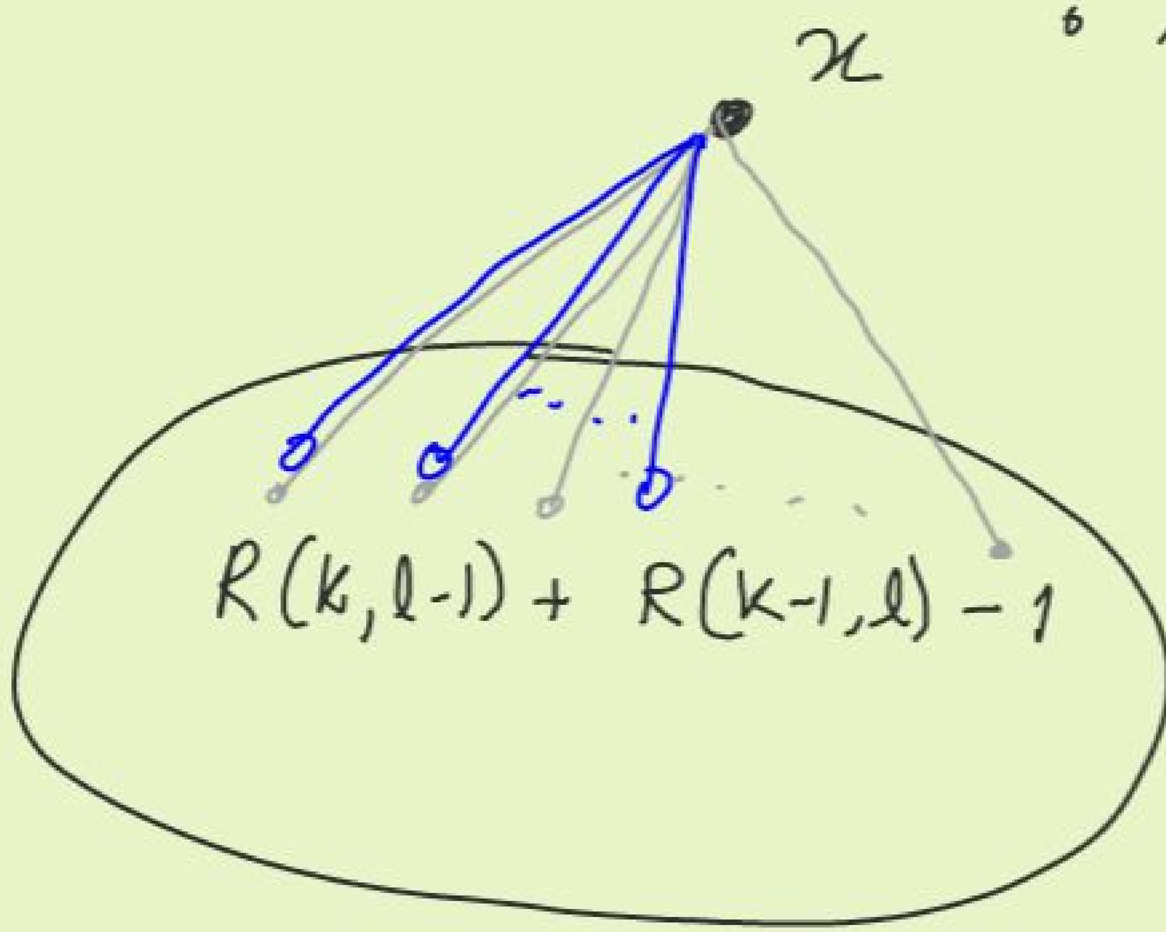
- $\ge R(k-1,l)$ red edges
- $\therefore$ On the end points there is either an l-clique of blue color

Proof of $R(k,l) \leq \binom{k+l-2}{k-1}$



- $\geq R(k-1,l)$ red edges
- $\therefore$ On the end points there is either an l-clique of blue color
- or a $(k-1)$-clique of red color which along with edges to x forms a k-clique of red color.

Proof of $R(k,l) \le \binom{k+l-2}{k-1}$

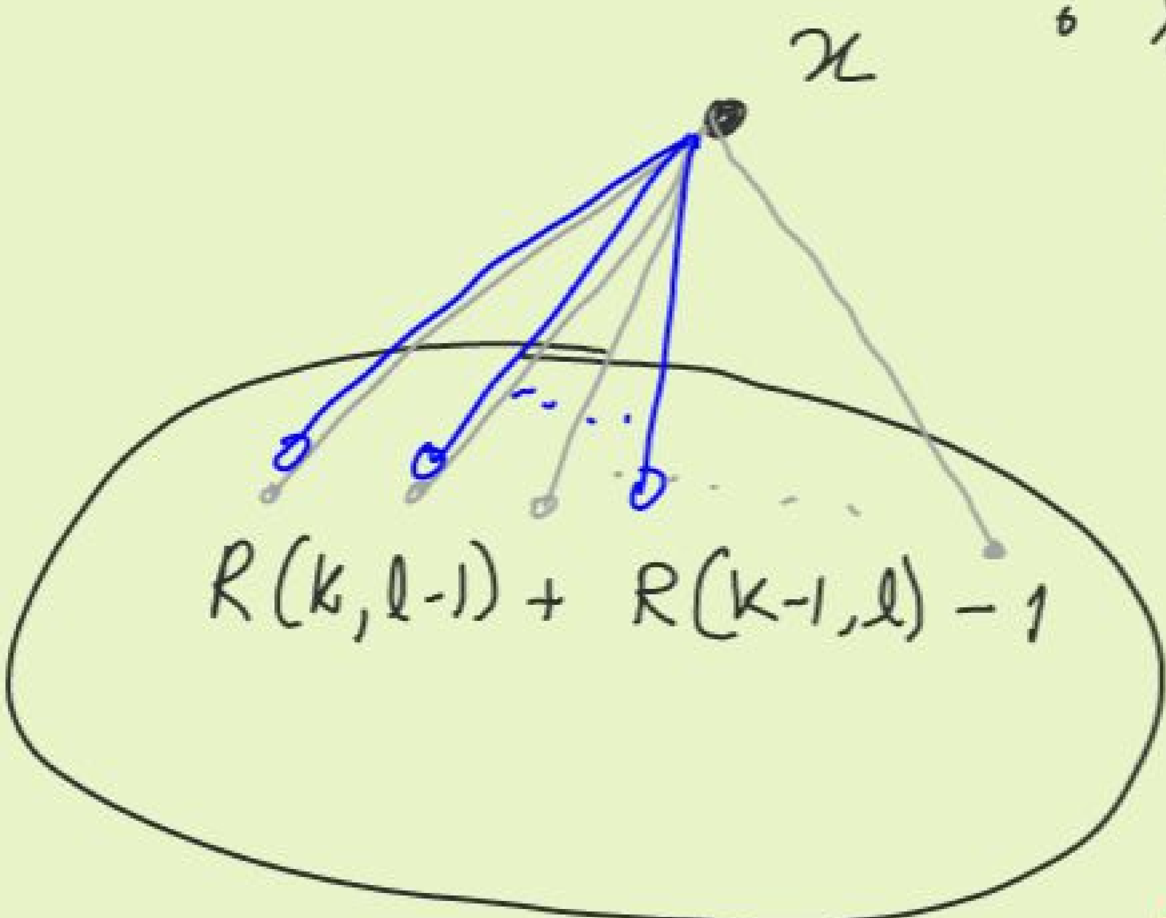


- similarly if $\geq R(k,l-1)$ blue edges

$$\therefore \; R(k,l) \le R(k,l-1) + R(k-1,l)$$

$$\le \binom{k+l-3}{k-1} + \binom{k+l-3}{k-2} = \binom{k+l-2}{k-1}$$

Proof of $R(k,l) \le \binom{k+l-2}{k-1}$



- similarly if $\geq R(k,l-1)$ blue edges

Base case of induction

$$\therefore \; R(k,l) \le R(k,l-1) + R(k-1,l)$$
$$\le \binom{k+l-3}{k-1} + \binom{k+l-3}{k-2} = \binom{k+l-2}{k-1}$$

Today:

- Probabilistic method.
- Starting with Module - 3

Ramsey numbers :

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

$$R(l,l) \leq \binom{2l-2}{l-1} \leq \frac{2^{2(l-2)}}{(2l-1)}$$

Ramsey numbers:

$$R(k,l) \le \binom{k+l-2}{k-1}$$

$$R(l,l) \le \underbrace{\binom{2l-2}{l-1} \le \frac{2^{2(l-2)}}{(2l-1)}}_{\text{why?}}$$

Ramsey numbers :

$$R(k,l) \le \binom{k+l-2}{k-1}$$

$$R(l,l) \le \binom{2l-2}{l-1} \le \frac{2^{2(l-2)}}{(2l-1)}$$

Today : $\boxed{R(l,l) \ge 2^{(l-2)/2}}$

$$\boxed{\text{Theorem [Erdös]:} \quad R(l,l) \geq 2^{(l-2)/2}}$$

<u>proof</u>: (By probabilistic method)

pick a random coloring:

i.e. each edge is colored

blue w.p. $1/2$

red w.p. $1/2$

$$\boxed{\text{Theorem [Erdös]: } R(\ell,\ell) \geq 2^{(\ell-2)/2}}$$

proof: (By probabilistic method)

pick a random coloring:

let X be a collection of ℓ vertices

$$\text{Prob}\left[\begin{array}{l}\text{all edges in } X \text{ are colored red or} \\ \text{all edges in } X \text{ are colored blue}\end{array}\right]$$

Theorem [Erdös]: $R(l,l) \geq 2^{(l-2)/2}$

proof: (By probabilistic method)

pick a random coloring:

let X be a collection of l vertices

$$\text{Prob}\begin{bmatrix} \text{all edges in X are colored red or} \\ \text{all edges in X are colored blue} \end{bmatrix}$$

$$= \frac{2}{2^{\binom{l}{2}}}$$

Theorem [Erdös]: $R(\ell,\ell) \geq 2^{(\ell-2)/2}$

<u>proof</u>: (By probabilistic method)

pick a random coloring:

$\therefore$

$$\text{Prob}\left[\begin{array}{l}\exists X:\ \text{all edges in } X \text{ are colored red or} \\ \qquad \text{all edges in } X \text{ are colored blue}\end{array}\right]$$

$$\leq \binom{n}{\ell} \cdot \frac{2}{2^{\binom{\ell}{2}}}$$

$\boxed{\text{Theorem [Erdös]: } R(\ell,\ell) \geq 2^{(\ell-2)/2}}$

proof: (By probabilistic method)

pick a random coloring:

$\because$

$$\text{Prob}\left[\exists X: \begin{array}{l}\text{all edges in } X \text{ are colored red or} \\ \text{all edges in } X \text{ are colored blue}\end{array}\right]$$

$$\leq \binom{n}{\ell} \cdot \frac{2}{2^{\binom{\ell}{2}}} \leq n^{\ell} \cdot 2^{1-\binom{\ell}{2}}$$

Suppose

$$\text{Prob}\left[\exists X : |X| = \ell,\ \begin{array}{l}\text{either } X \text{ is a blue clique} \\ \text{or } X \text{ is a red clique}\end{array}\right] < 1$$

Suppose

$$\text{Prob}\left[\exists X : |X| = l, \begin{array}{l} \text{either } X \text{ is a blue clique} \\ \text{or } X \text{ is a red clique} \end{array}\right] < 1$$

$$\Downarrow$$

$$\text{Prob}\left[\forall X : |X| = l, \begin{array}{l} \text{neither } X \text{ is a blue clique} \\ \text{nor } X \text{ is a red clique} \end{array}\right] > 0$$

Suppose

$$\text{Prob}\left[\exists X : |X| = \ell, \begin{array}{l}\text{either } X \text{ is a blue clique} \\ \text{or } X \text{ is a red clique}\end{array}\right] < 1$$

$$\Downarrow$$

$$\text{Prob}\left[\forall X : |X| = \ell, \begin{array}{l}\text{neither } X \text{ is a blue clique} \\ \text{nor } X \text{ is a red clique}\end{array}\right] > 0$$

$$\Downarrow$$

$\exists$ a coloring of edges such that there is no monochromatic ℓ-clique.

Theorem [Erdös]: $R(\ell,\ell) \geq 2^{(\ell-2)/2}$

<u>proof</u>: (By probabilistic method)

pick a random coloring:

$$\text{Prob}\left[\begin{array}{l}\exists X: \text{all edges in } X \text{ are colored red or} \\ \qquad \text{all edges in } X \text{ are colored blue}\end{array}\right] \leq n^{\ell}\, 2^{1-\binom{\ell}{2}} < n^{\ell} \cdot 2^{-\frac{\ell(\ell-2)}{2}}$$

Theorem [Erdös]: $R(\ell,\ell) \geq 2^{(\ell-2)/2}$

<u>proof</u>: (By probabilistic method)

pick a random coloring:

$$\text{Prob}\left[\begin{array}{l}\exists X: \text{ all edges in } X \text{ are colored red or} \\ \text{ all edges in } X \text{ are colored blue}\end{array}\right] \leq n^{\ell} 2^{1-\binom{\ell}{2}} < n^{\ell} \cdot 2^{-\frac{\ell(\ell-2)}{2}}$$

i.e. $n^{\ell} < 2^{\ell(\ell-2)/2}$

$n < 2^{(\ell-2)/2}$

Theorem [Erdös]: $R(\ell,\ell) \geq 2^{(\ell-2)/2}$

proof: Putting it together:

if $n < 2^{(\ell-2)/2}$ then $\exists$ a coloring of edges (into 2 colors)

such that

neither a red ℓ-clique

nor a blue ℓ-clique

$\boxed{\text{Theorem [Erdös]: } R(l,l) \geq 2^{(l-2)/2}}$

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$\square$

A bit about probabilistic method:

- technique first discovered & effectively used by Paul Erdös

Module - 3

Graph Theory

A graph $G = (V, E)$ is called bipartite if the vertices of the graph can be partitioned $V = X \dot{\cup} Y$ s.t.

$\forall e = (u, v) \in E$: either $u \in X$ and $v \in Y$

or $u \in Y$ and $v \in X$.

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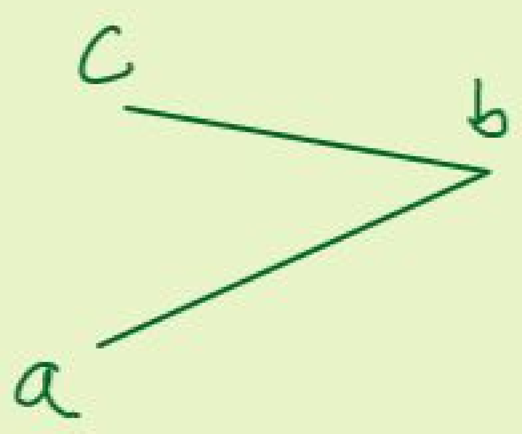
Example :



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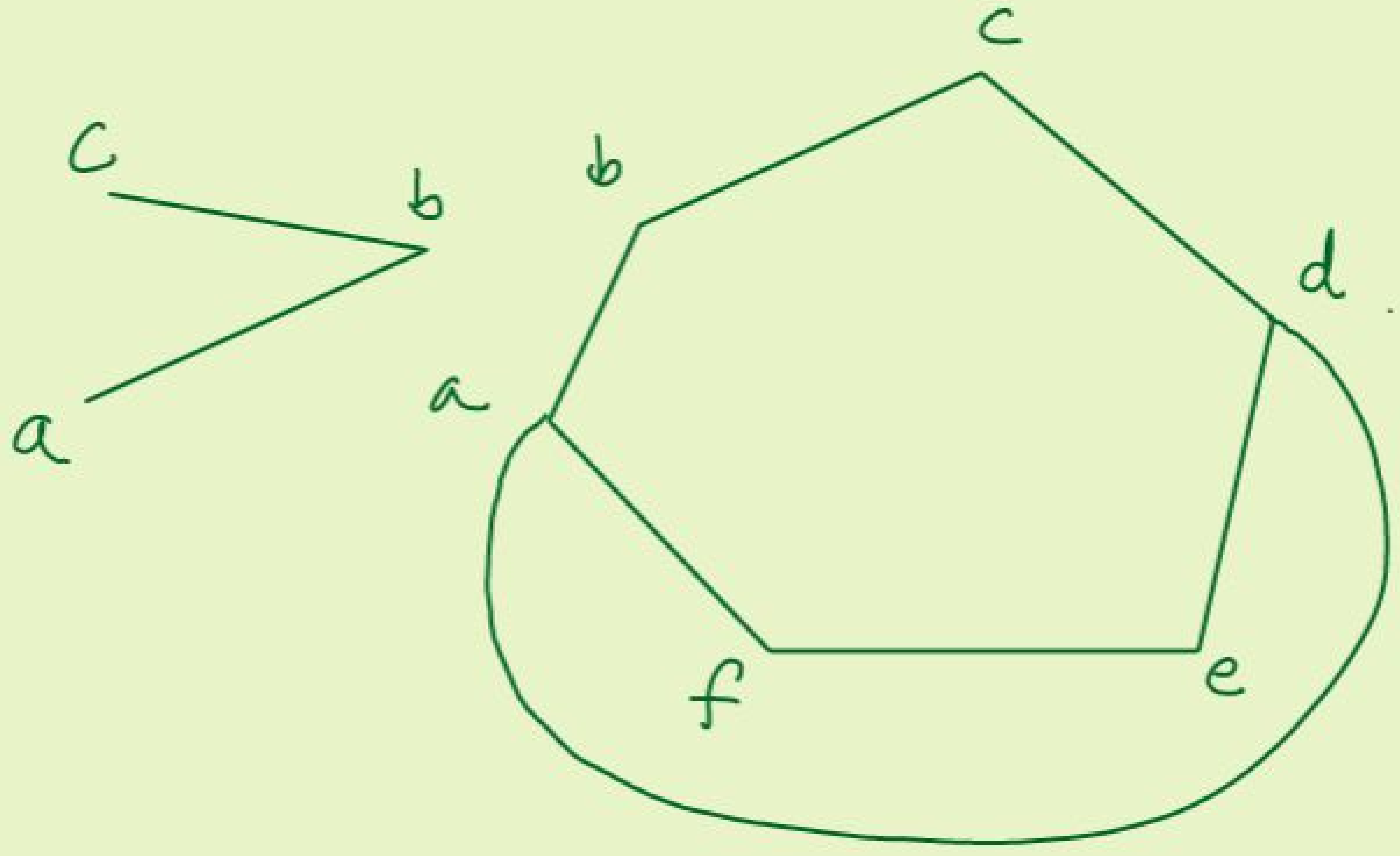

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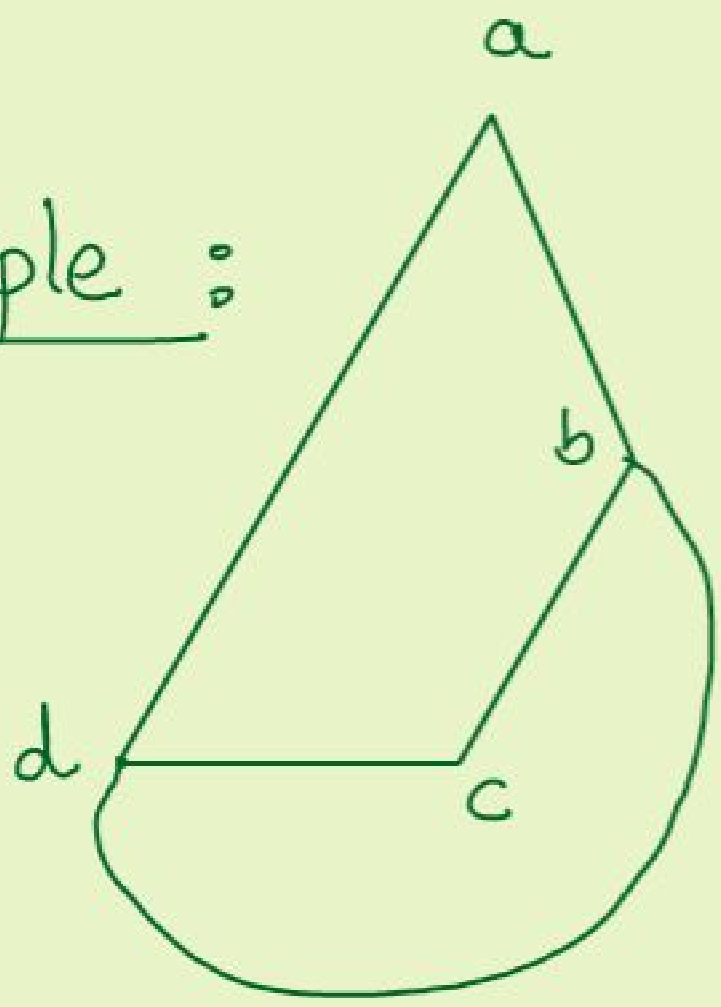

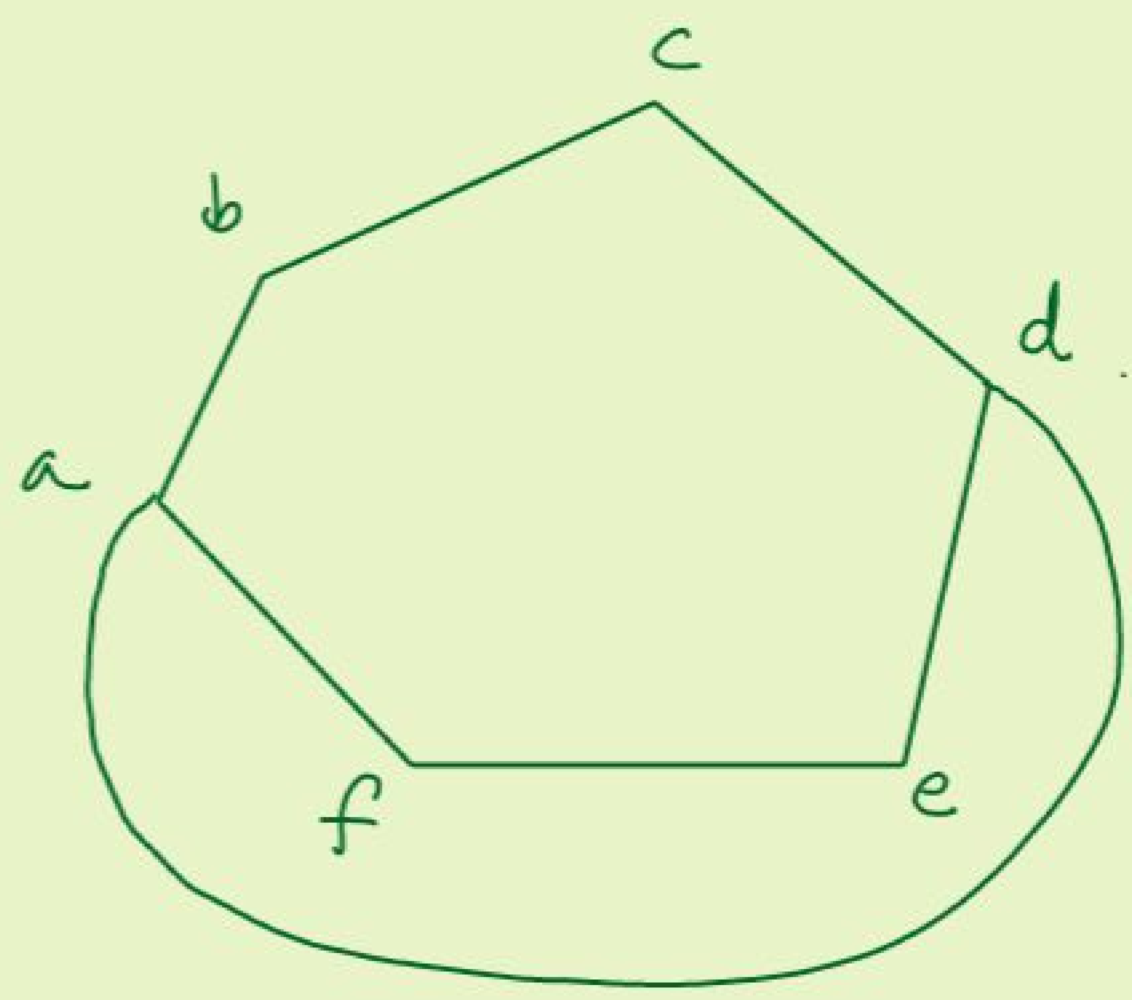

Theorem: A graph is bipartite if and only if it does not have an odd cycle.

Some definitions: Let $G = (V, E)$ be a graph:

Walk: is a sequence of vertices $v_1, v_2, v_3 \dots v_n$ such that $\forall\, i \in [n-1]$, $(v_i, v_{i+1}) \in E$.

Path: a walk is a path if no vertex repeats itself.

Closed walk: a walk is called a closed walk if it starts and ends at the same vertex.

Cycle: a closed walk $v_1, v_2, \dots, v_\ell, v_1$ is called a cycle if $v_1 \neq v_2 \neq \dots \neq v_\ell$.