# CS 207

# Discrete Structures

Nutan Limaye

21 OCT 2013

# Module 4

## Abstract Algebra.

Last few classes:

Module 3 (Graph Theory)

- bipartite graphs
- hall's condition
- stable matchings
- computing maximum matchings
- tournament graphs
- dominating sets. . . .

Today:

Group theory

- definition of groups
- examples of groups
- applications of group theory.
- properties of groups.

A set $S$ along with an operator $*$, $(S,*)$, is called a group if the operator satisfies the following properties wrt to the elements of $S$:

Closure : $\forall a,b \in S \quad a*b \in S$

Associativity : $\forall a,b,c \in S \quad a*(b*c) = (a*b)*c$

Identity : $\exists e \in S$ s.t. $\forall a \in S \quad a*e = e*a = a$

Inverse : $\forall a \in S \ \exists a' \in S$ s.t. $a*a' = e$

Examples

| | Group | Not a Group |
|---|---|---|
| 1. $(\mathbb{Z}_p, \times_p)$ | | |
| 2. $(\mathbb{Z}_p, +_p)$ | | |
| 3. $(\mathbb{Z}_p^*, \times_p)$ | | |

$p$ : a prime

$\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$

$\times_p :=$ multiplicatn mod $p$

$+_p :=$ addition mod $p$

$\mathbb{Z}_p^* := \mathbb{Z}_p \setminus \{0\}$

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Application of Group Theory (Cryptography).

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Akbar

# Application of Group Theory (Cryptography).

Akbar

Birbal.

Application of Group Theory (Cryptography).

Akbar

(from Mumbai)

$\longrightarrow$

$\longleftarrow$

Birbal.

(from Chennai)

⋮

Trying to communicate over an insecure channel.

- Share a secrete key
- communicate using the shared secrete.

- But how can they share a secrete securely?

Akbar Birbal.

Akbar

Birbal.

Choose $p$ a prime,

$g$ an element from $\mathbb{Z}_p$

let $\alpha$ be a private key

let $A \leftarrow g^{\alpha} \bmod p$

Akbar Birbal.

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$$\xrightarrow{g,\, p,\, A}$$

| | |
|---|---|
| | let $\beta$ be private key |
| | let $B \leftarrow g^{\beta} \bmod p$ |

Akbar | Birbal.

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Akbar | Birbal

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$\xleftarrow{B}$

compute $B^{a} \bmod p$

Akbar

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Compute $A^{\beta} \bmod p$

!! Shared Secrete $g^{\alpha\beta} \bmod p$ !!

Simple properties of groups

- A group has a unique identity.

## Simple properties of groups

$\boxed{(S, *) \text{ is a group}}$

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$$e_2*e_1 = e_1*e_2 = e_2$$

$$e_1*e_2 = e_2*e_1 = e_1$$

$$\therefore \quad e_1 = e_2 \quad \Rightarrow\Leftarrow$$

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Suppose $\exists a$: $a_1^{-1}, a_2^{-1}$ are its two distinct inverses

$$a * a_1^{-1} = e \Rightarrow a * a_1^{-1} * a_1 = e * a_1 \Rightarrow a = a_1$$

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$\therefore a_1 = a_2$

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- For any two elements $(a*b)^{-1} = b^{-1} * a^{-1}$

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$$(a*b) * (a*b)^{-1} = e$$

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$$(a*b) * (a*b)^{-1} = e$$

$$\Rightarrow \quad a^{-1} * (a*b) * (a*b)^{-1} = a^{-1} * e$$

$$\Rightarrow \quad (a^{-1} * a) * b * (a*b)^{-1} = a^{-1} \qquad \text{(Associativity, identity)}$$

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$$\Rightarrow \quad e * b * (a*b)^{-1} = a^{-1} \qquad \text{(Inverse)}$$

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$$(a*b)*(a*b)^{-1} = e$$

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$$\Rightarrow \quad (a^{-1}*a)*b*(a*b)^{-1} = a^{-1} \qquad \text{(Associativity, identity)}$$

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CS 207

Discrete Structures

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24 OCT 2013

# Module 4

## Abstract Algebra.

## Last Time

- Definition of groups
- Examples of groups
- Application of group theory.
- Simple properties of groups

Today:

- More examples of groups
- Abelian and non-abelian groups
- Subgroups & simple properties of subgroups
- Another application of groups.

A set $S$ along with an operator $*$, $(S,*)$, is called a group if the operator satisfies the following properties wrt to the elements of $S$:

Closure : $\forall a,b \in S \quad a*b \in S$

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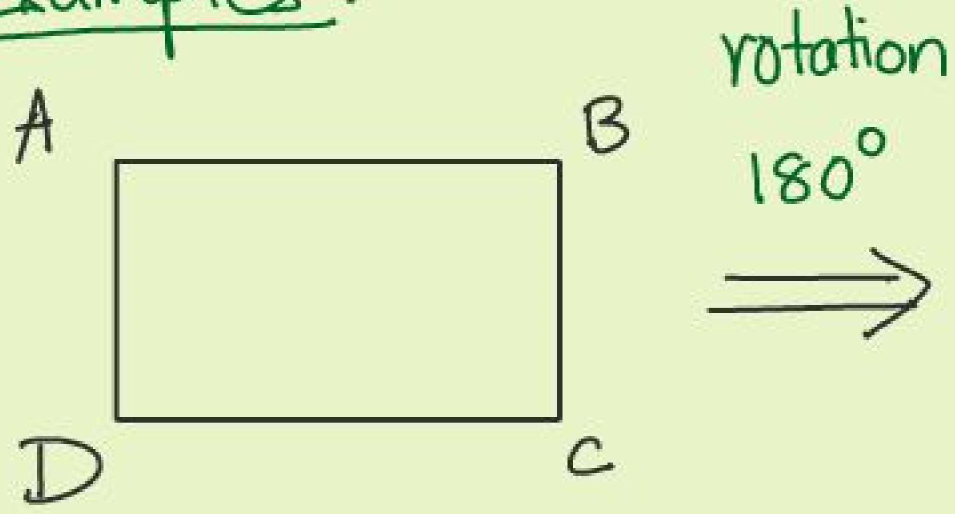
A rigid motion of an object which maps the object to itself is called a <u>symmetry</u>

<u>Examples</u>:



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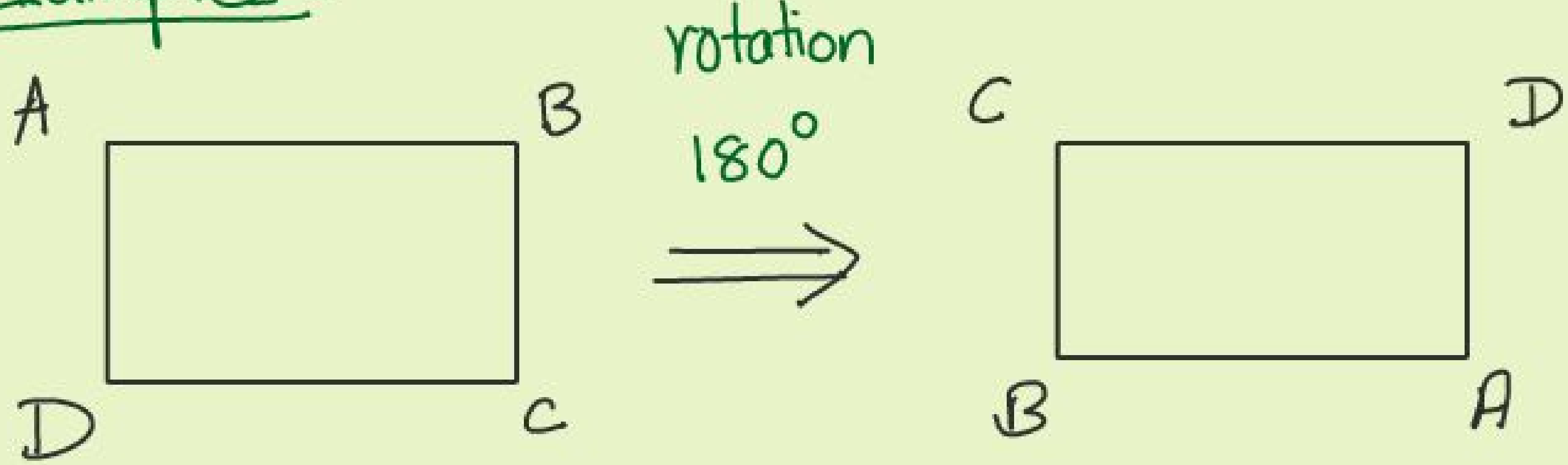

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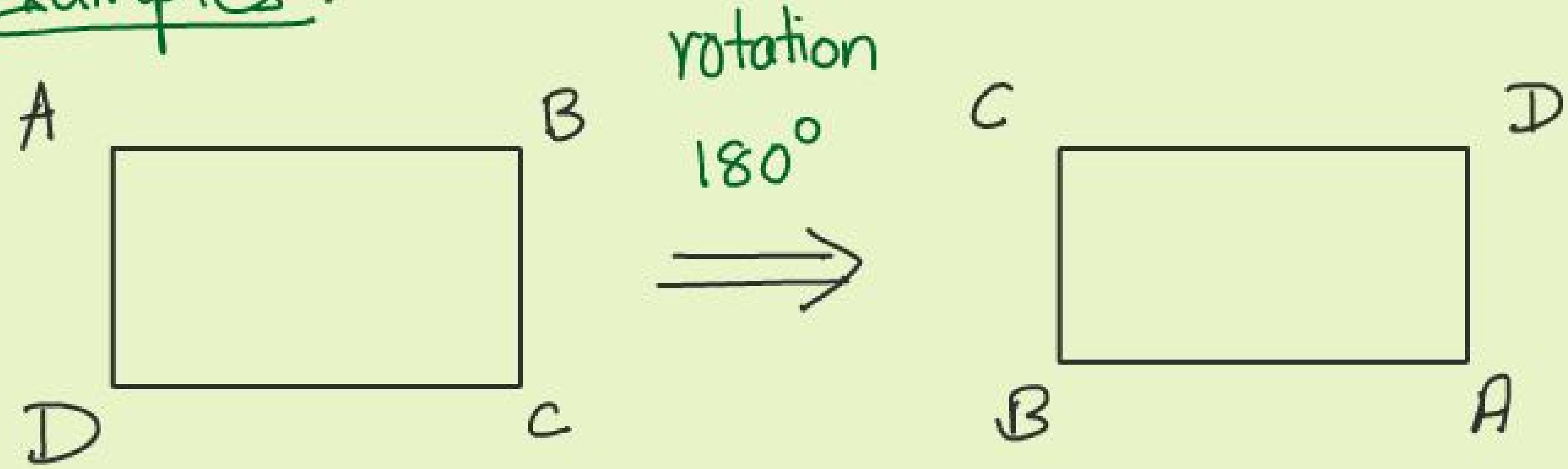


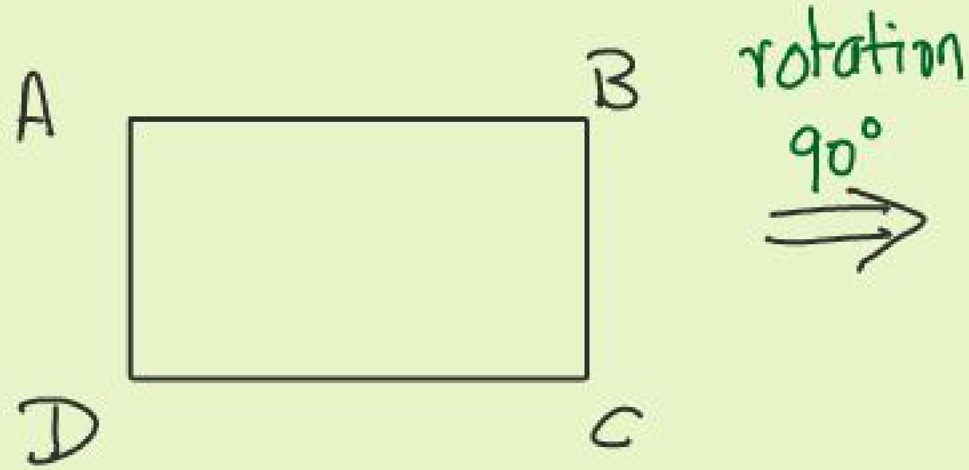

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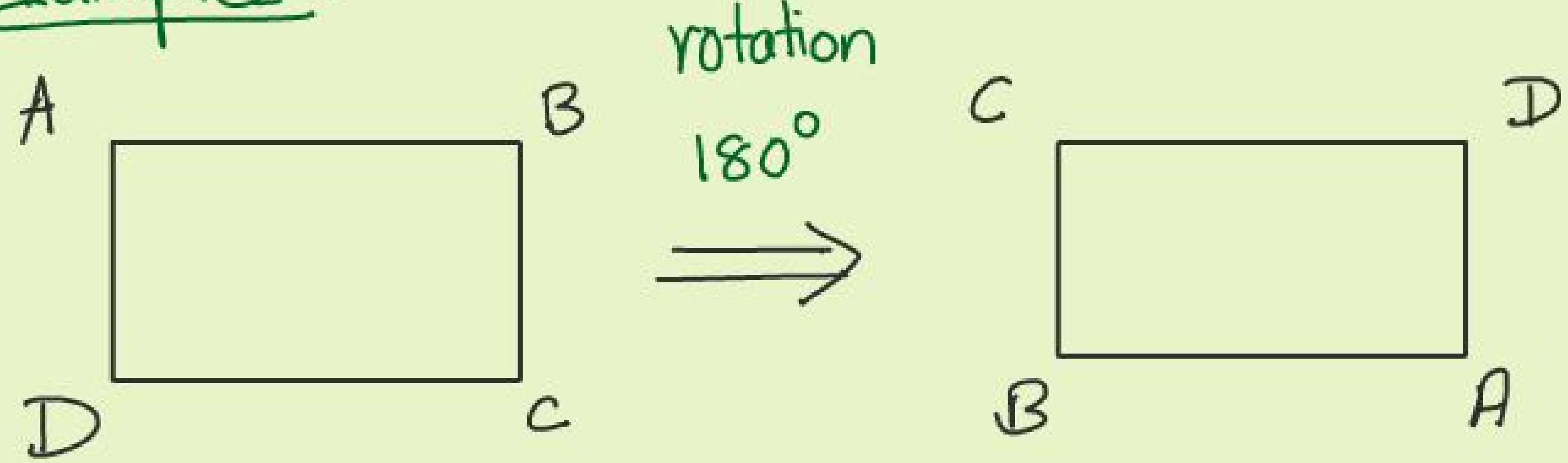


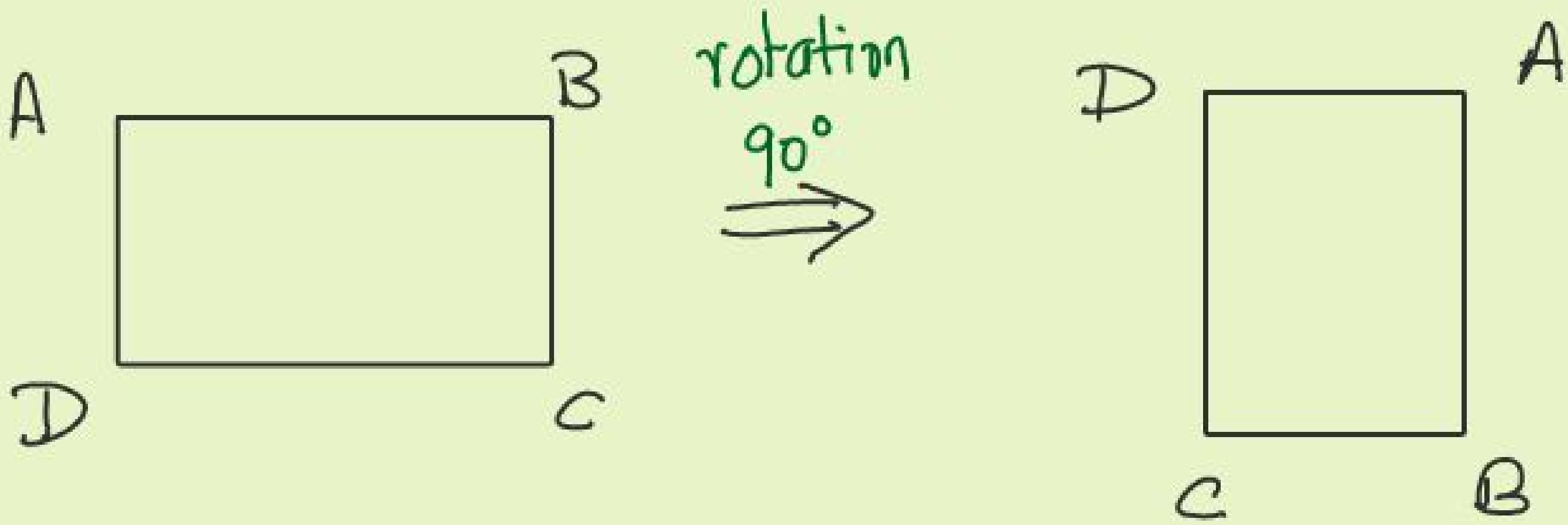

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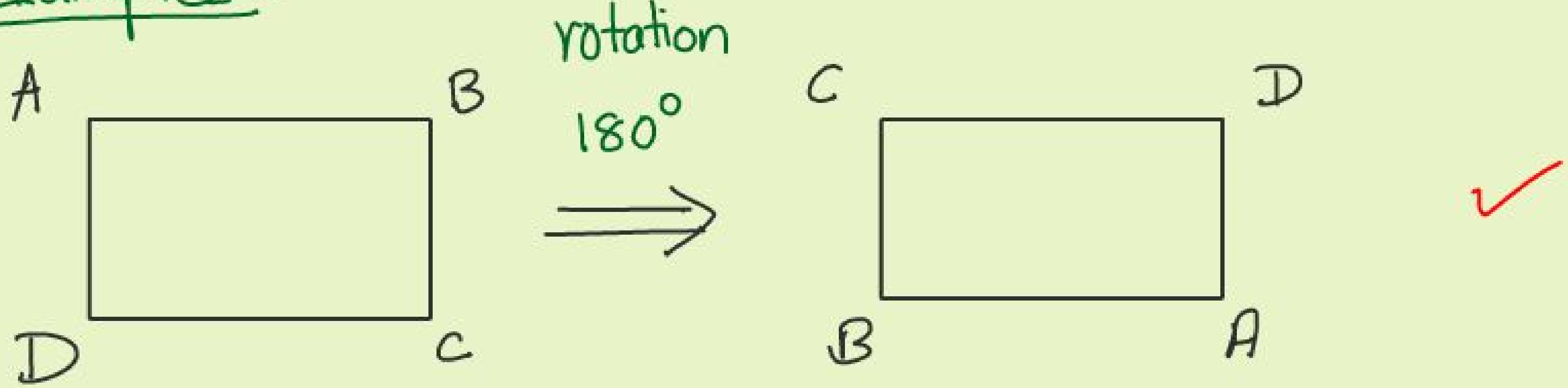


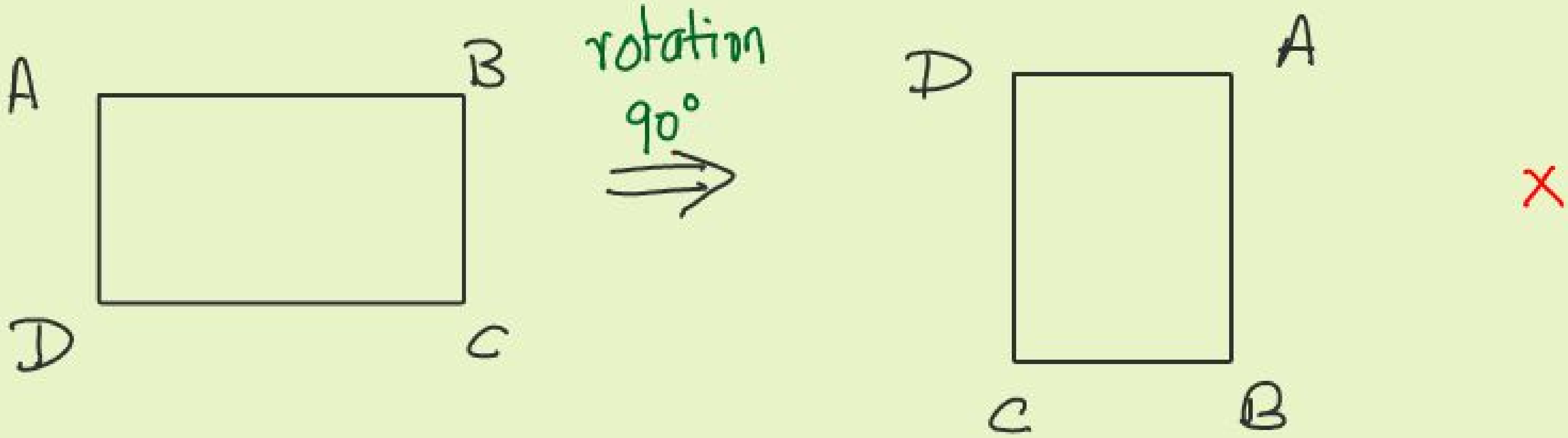

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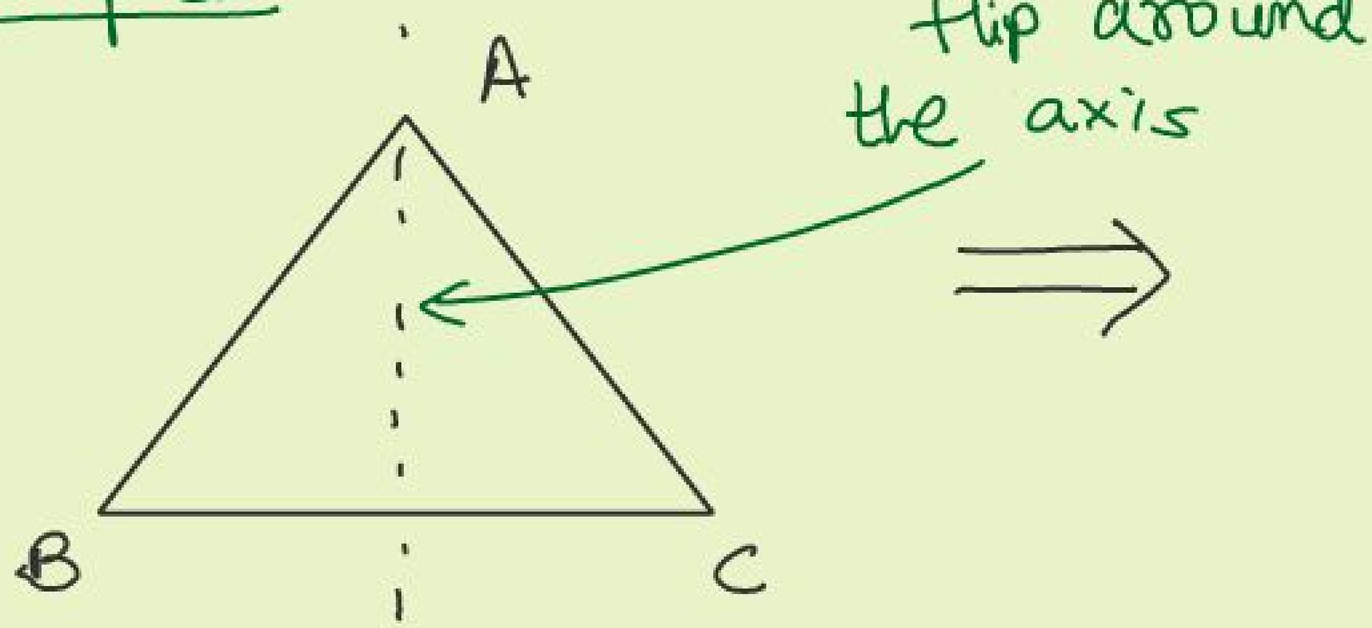


$\Longrightarrow$

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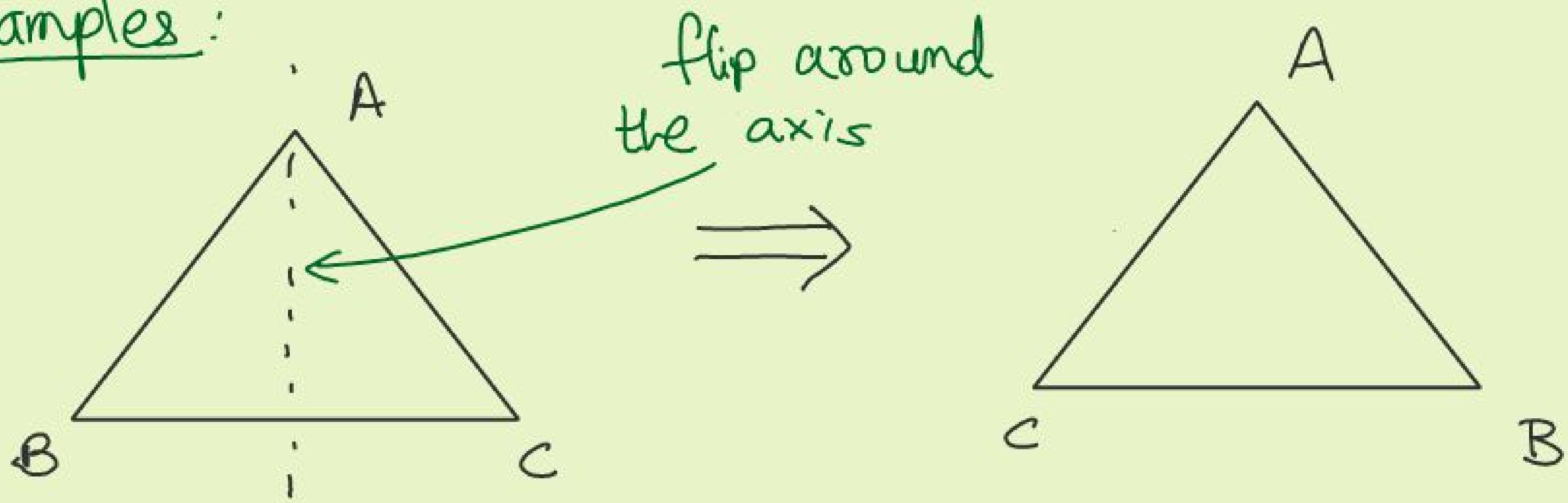

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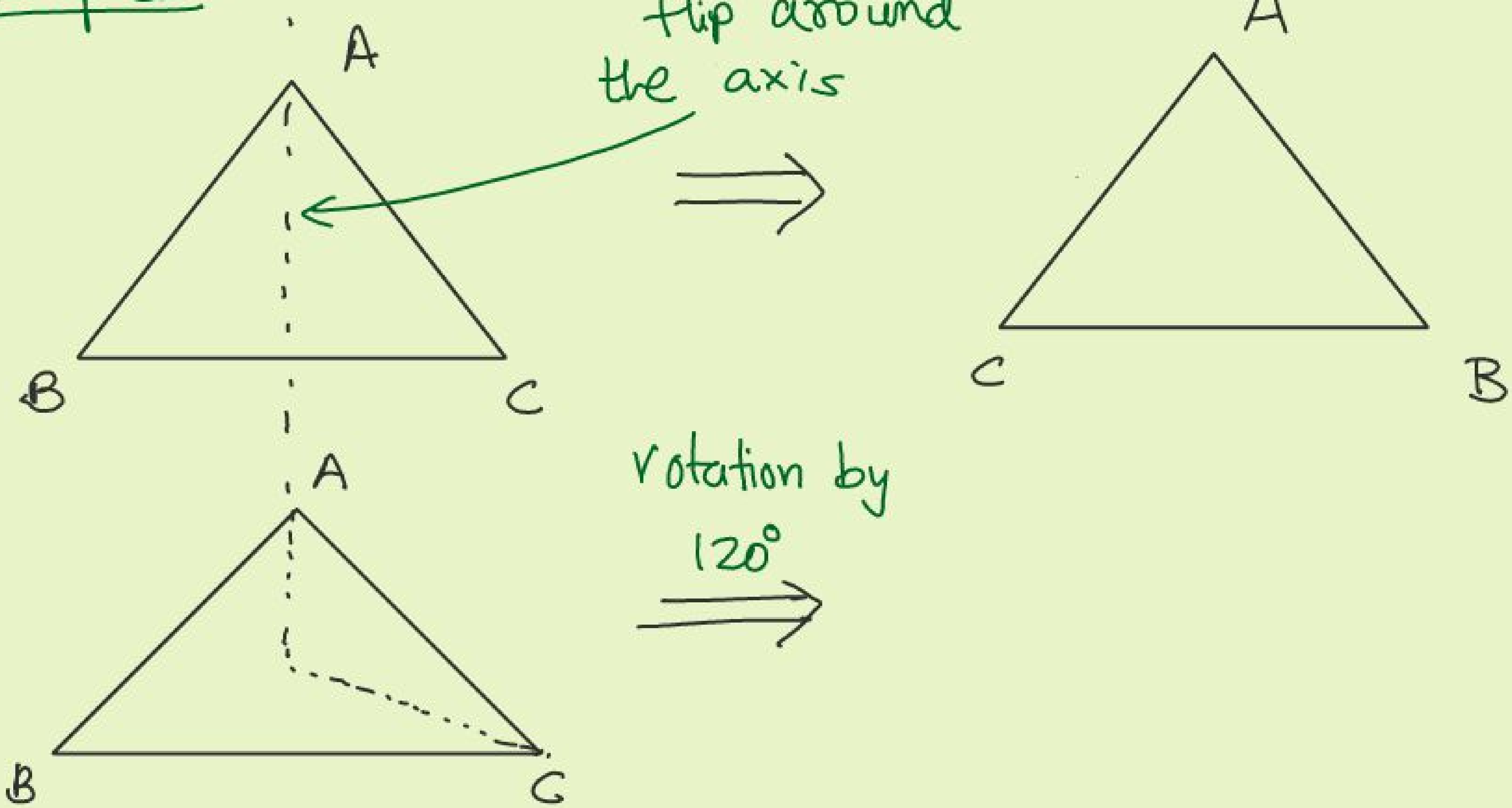

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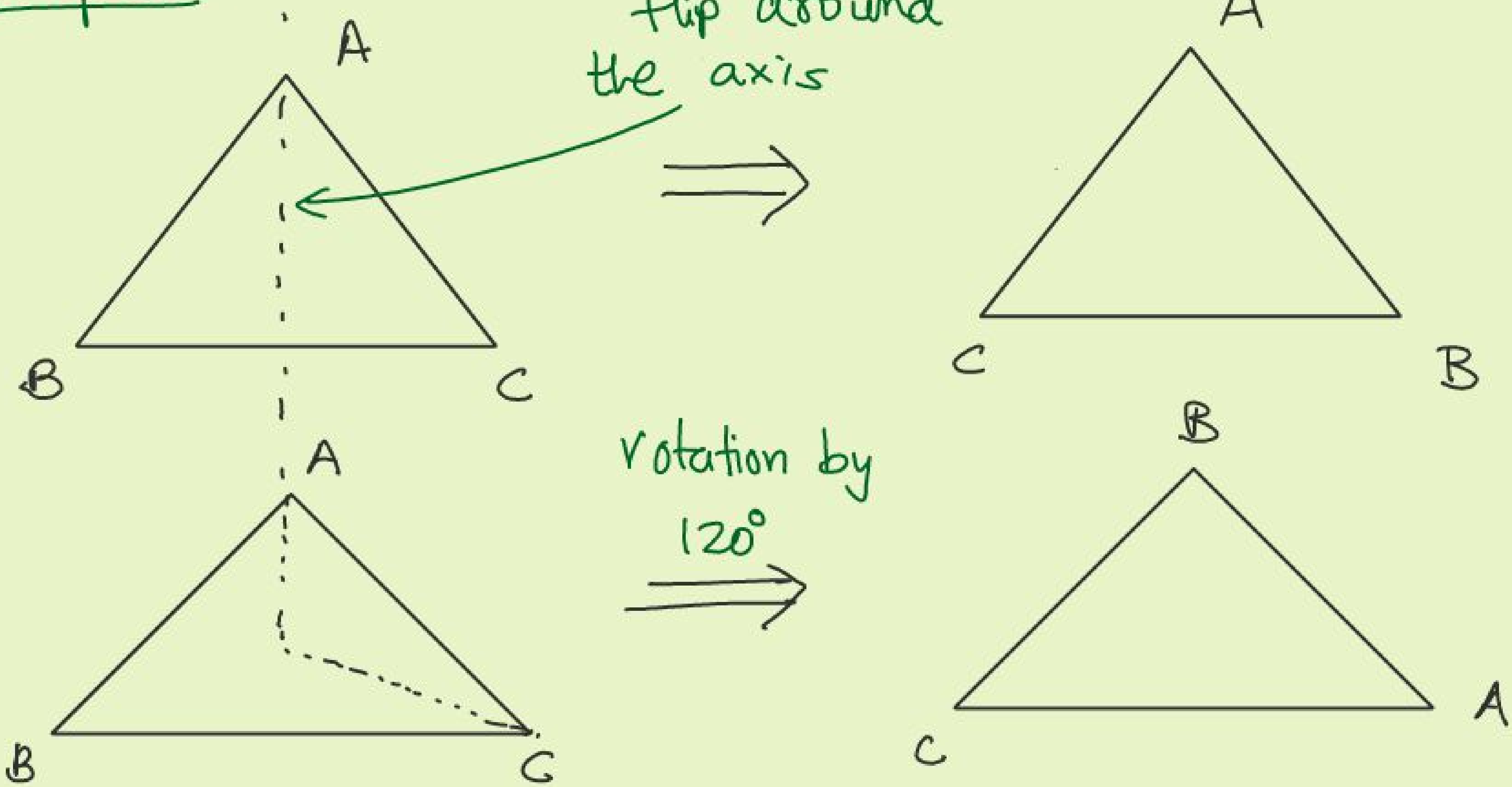

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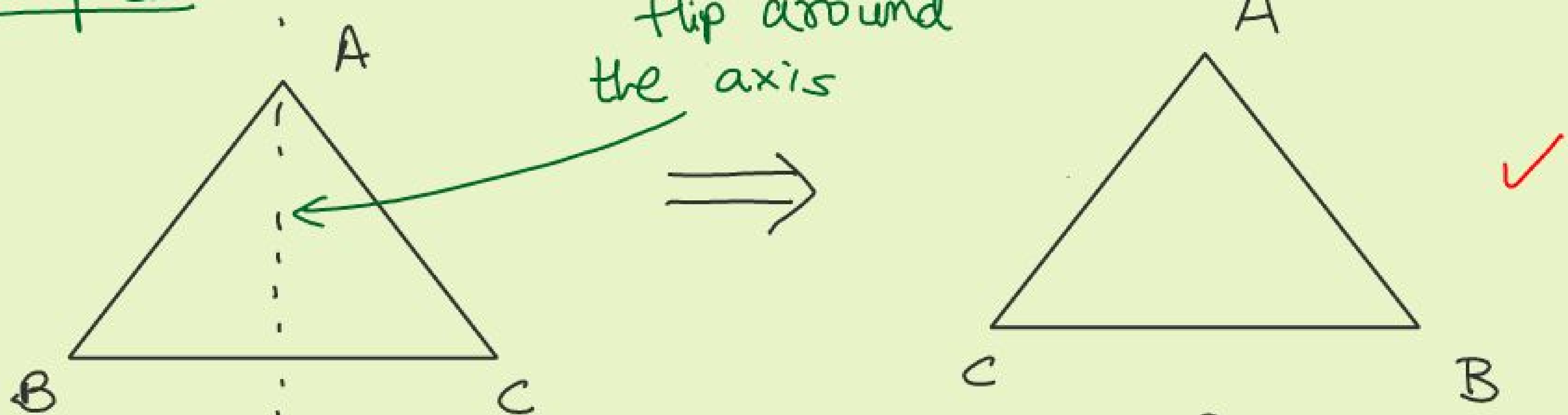


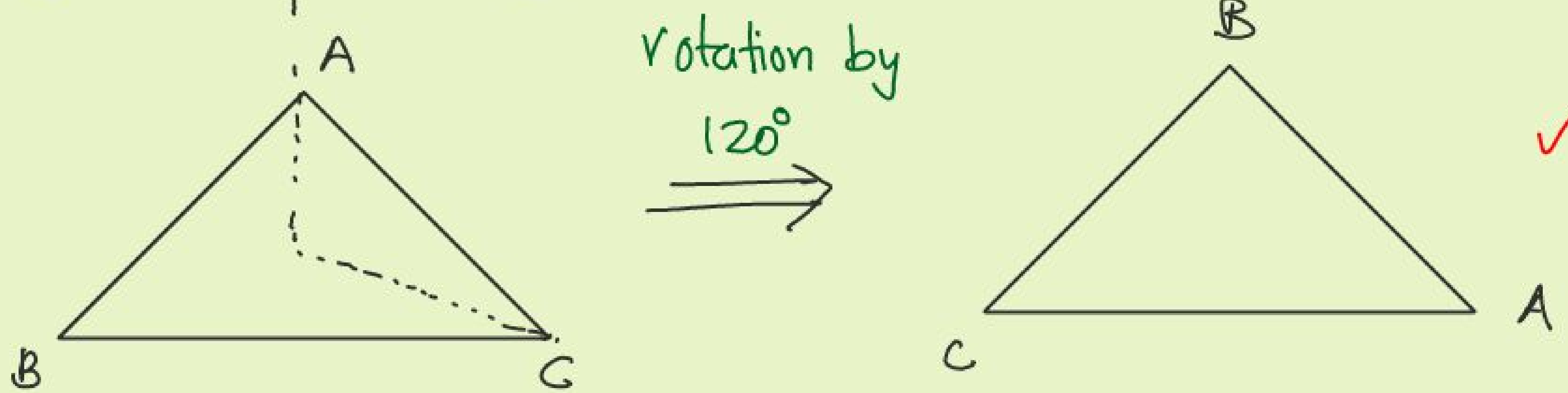

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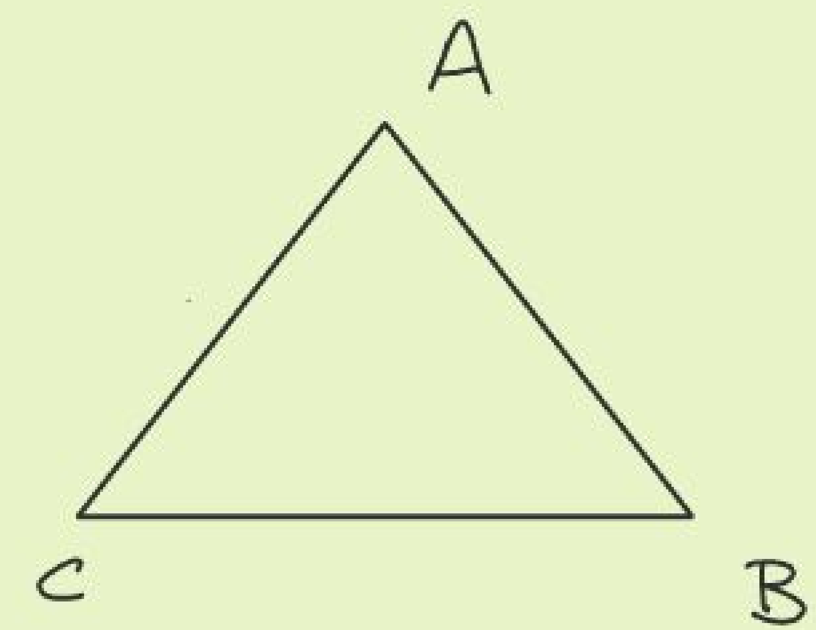
Examples:

- How many distinct symmetries does an equilateral triangle have?



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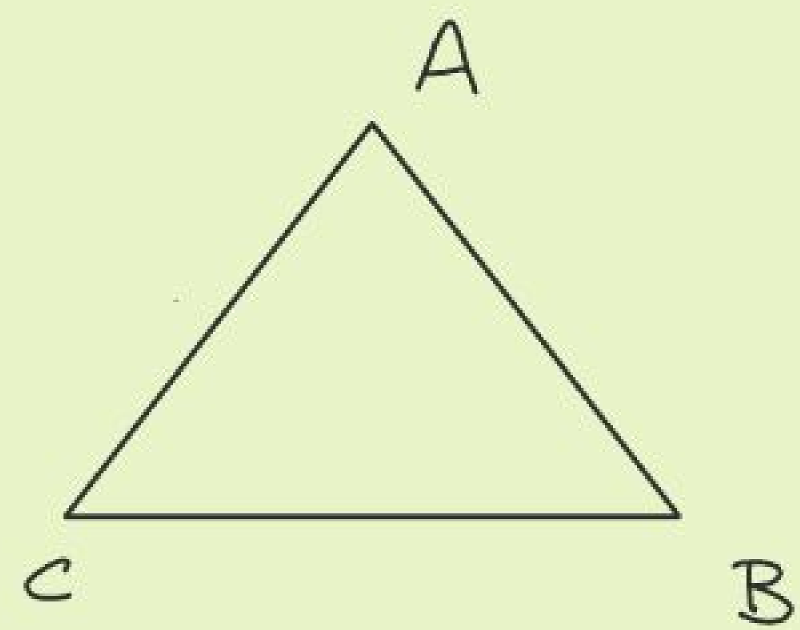


- What is a composition of two symmetries?

# Examples of groups

1. $S :=$ Symmetries of an equilateral triangle
   $* :=$ composition of symmetries

$(S, *)$ as defined above forms a group

- closure
- associativity of $*$
- identity
- inverse.

## Examples of groups

2. $S_n =$ all permutations of $\{1, 2, \dots, n\}$

$*$ = composition of permutations

$(S_n, *)$ defined as above forms a group.

## Examples of groups

- Let $S = M_2(\mathbb{R}) \leftarrow$ set of all $2\times 2$ matrices over $\mathbb{R}$.

# Examples of groups

- Let $S = M_2(\mathbb{R})$ ← set of all $2\times 2$ matrices over $\mathbb{R}$.

  $*$ = matrix multiplication.

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- Let $S = M_2(\mathbb{R}) \setminus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

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- Let $S = GL_2(\mathbb{R})$ ← set of all invertible matrices over $\mathbb{R}$

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- Let $S = GL_2(\mathbb{R})$ ← set of all invertible matrices over $\mathbb{R}$

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  $(S,*)$ defined as above forms a group ✓

A group $(S, *)$ is called <u>abelian</u> if $\forall a, b \in S$

$$a * b = b * a.$$

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## Examples

Abelian

$(\mathbb{Z}_p, +_p)$

$(\mathbb{Z}_p, \times_p)$

Non-abelian

?

Let $G = (S, *)$ be a group. Let $T \subseteq S$

$H = (T, *)$ is called a <u>subgroup</u> of $G$ if $H$ is a group.

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$H = (T, *)$ is called a **subgroup** of $G$ if $H$ is a group.

## Examples of subgroups :

1. Let $SL_2(\mathbb{R})$ be all $2 \times 2$ matrices with determinant 1.

   $(SL_2(\mathbb{R}), \times)$ is a subgroup of $(GL_2(\mathbb{R}), \times)$

Let $G = (S, *)$ be a group. Let $T \subseteq S$

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## Examples of subgroups :

1. Let $SL_2(\mathbb{R})$ be all $2 \times 2$ matrices with determinant 1.

   $(SL_2(\mathbb{R}), \times)$ is a subgroup of $(GL_2(\mathbb{R}), \times)$

2. Let $G = (S, *)$ be a group. Let $a \in S$.

   Let $\langle a \rangle = \{ a^k \mid k \in \mathbb{Z} \}$

   $(\langle a \rangle, *)$ is a subgroup of $G$.

CS 207

Discrete Structures

Nutan Limaye

28 OCT 2013

# Module 4

## Abstract Algebra.

## Last time

- Examples of groups
- Abelian and non-abelian groups
- Subgroups & examples of subgroups
  - cyclic subgroup

Today:

- Properties of subgroups
- Properties of cyclic subgroups.
- Cosets & Lagrange's theorem.

Let $G = (S, *)$ be a group. Let $T \subseteq S$. And let $H = (T, *)$. $H$ is a subgroup of $G$ iff.

- $T \neq \phi$
- whenever $g, h \in T$ then $gh^{-1} \in T$

Let $G = (S, *)$ be a group. Let $T \subseteq S$. And let $H = (T, *)$. $H$ is a subgroup of $G$ iff.

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<u>Proof</u>: $(\Rightarrow)$

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Proof: $(\Rightarrow)$ suppose $H$ is a subgroup of $G$. Then $T \neq \emptyset$

Now suppose $g, h \in T$ Then $h^{-1} \in T$ $\therefore gh^{-1} \in T$

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<u>Proof</u> : ($\Rightarrow$) suppose $H$ is a subgroup of $G$. Then $T \neq \phi$

Now suppose $g, h \in T$ Then $h^{-1} \in T$ $\therefore gh^{-1} \in T$

($\Leftarrow$) Suppose $T = \phi$ and $\forall g, h \in T$ $gh^{-1} \in T$

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<u>Proof</u>: ($\Rightarrow$) suppose $H$ is a subgroup of $G$. Then $T \neq \phi$

Now suppose $g, h \in T$ Then $h^{-1} \in T$ $\therefore gh^{-1} \in T$

($\Leftarrow$) Suppose $T = \phi$ and $\forall g, h \in T$ $gh^{-1} \in T$

- Then for $g = h$ $g \cdot g^{-1} = e$ is in $T$.

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- $T \neq \phi$
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<u>Proof</u>: $(\Rightarrow)$ suppose $H$ is a subgroup of $G$. Then $T \neq \phi$

Now suppose $g, h \in T$ Then $h^{-1} \in T$ $\therefore gh^{-1} \in T$

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$\because$ $a^k \in T$ and $h^{-q} \in T$ $\Rightarrow$ $a^r \in T$

But $r < s$ $\Rightarrow\Leftarrow$

CS 207

Discrete Structures

Nutan Limaye

29 OCT 2013

# Module 4

## Abstract Algebra.

Last time :

- Properties of subgroups
- Properties of cyclic subgroups.

Today:

- Definition of cosets
- examples of cosets
- properties of cosets.

Let $G = (S, *)$ be a group and $H = (T, *)$ be its subgroup

A left coset of $G$ with representative $g \in S$ is defined to be

$$g \cdot H = \{ g * h \mid h \in T \}$$

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$g = 2 \qquad gH = \{ 2 +_{mod} 0, 2 +_{mod\,6} 3 \} = \{2, 3\}$

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Let $H$ be a subgroup of $G=(S,*)$. Let $g_1, g_2 \in S$. Then the following are equivalent:

(a) $g_1H = g_2H$, (b) $Hg_1^{-1} = Hg_2^{-1}$, (c) $g_2^{-1}g_1 \in H$, (d) $g_2 \in g_1H$

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<u>proof</u>: $[(a) \Rightarrow (c)]$ $g_1 H = g_2 H \Rightarrow g_2^{-1} g_1 H = H$.

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<u>proof</u>: [(a) $\Rightarrow$ (c)] $g_1 H = g_2 H \rightarrow g_2^{-1} g_1 H = H$.

$\Rightarrow g_2^{-1} g_1 \in H$.

[(c) $\Rightarrow$ (a)]

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<u>proof</u>: [(a) $\Rightarrow$ (c)] $g_1H = g_2H \implies g_2^{-1}g_1H = H$

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[(c) ⇒ (b)]

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[(c) $\Rightarrow$ (d)] $g_2^{-1}g_1 \in H \Rightarrow g_1^{-1}g_2 \in H$

$\Rightarrow g_1^{-1}g_2 = h$ for some $h \in H$

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All the left cosets of a group partition the group.

<u>proof</u>: if $g_1 \neq g_2$ then either $g_1 H = g_2 H$

or $g_1 H \cap g_2 H = \phi$

Suppose $g_1 = g_2$. . Let $f \in g_1 H \cap g_2 H$

$\therefore \exists h_1, h_2 \in H$ s.t. $g_1 h_1 = g_2 h_2 = f$

$g_2^{-1} g_1 h_1 = h_2$ i.e. $(g_2^{-1} g_1) \in h \Rightarrow g_1 H = g_2 H$

The same can be proved for right cosets.

let $G$ be a group and $H$ be its subgroup. The number of left cosets of $H$ in $G$ is the same as the number of right cosets of $H$ in $G$.

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And prove that it is a bijection.

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# CS 207

# Discrete Structures

Nutan Limaye

11 NOV 2013

# Module 4

## Abstract Algebra.

Today:

- Counting necklaces – Pólya's theory of counting

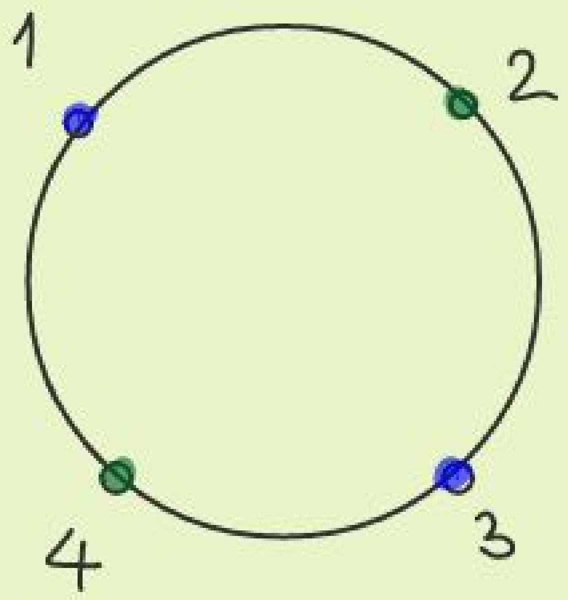
Given: $n$ beads of $m$ different colors $(m < n)$

Output: # different necklaces that can be formed?

<u>Recall</u>: Defn of

- groups
- symmetric groups
- cyclic subgroup of a symmetric group
- group of symmetries of a regular $n$-gon.

Consider the case when $n = 4$ and $m = 2$



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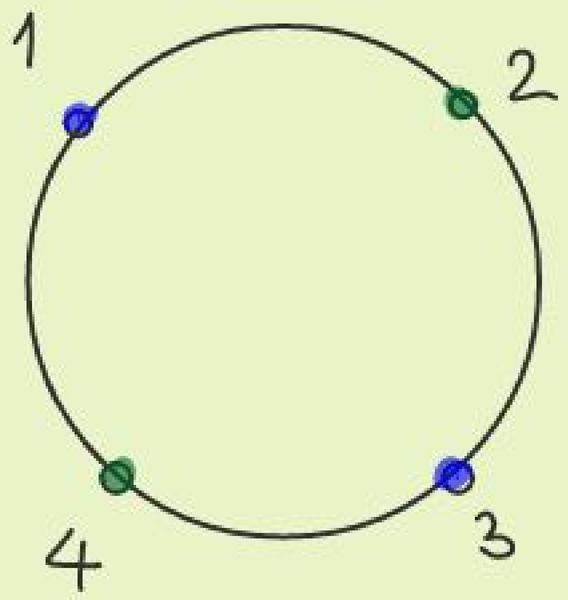


$S = \{1, 2, 3, 4\}$

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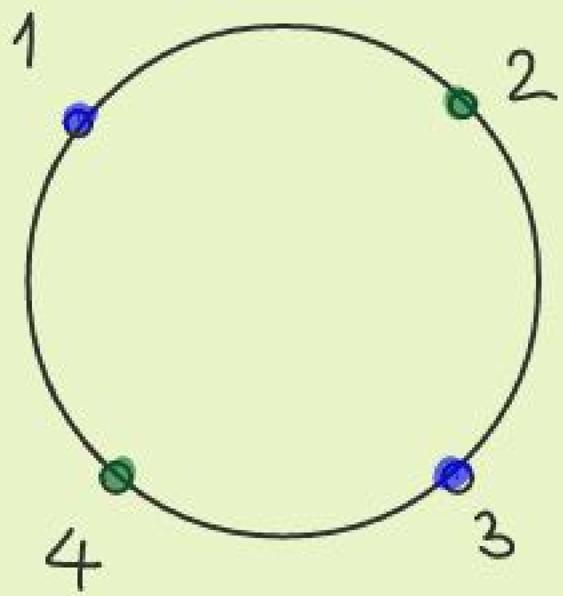


$S = \{1, 2, 3, 4\}$

$C = \{\text{set of all colorings}\}$

$= \{gggg, gggr, ggrg, \ldots\}$

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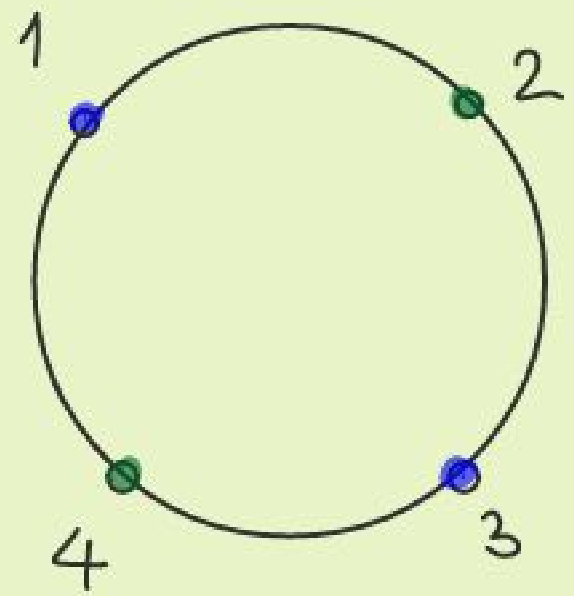


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Let $\pi = (12)(34)$

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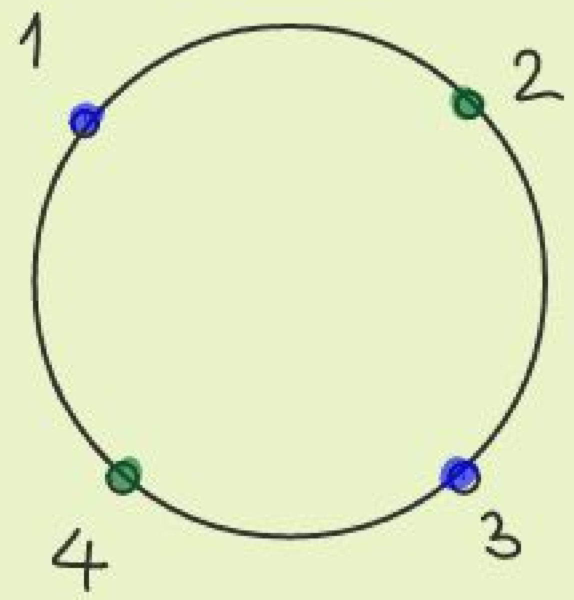


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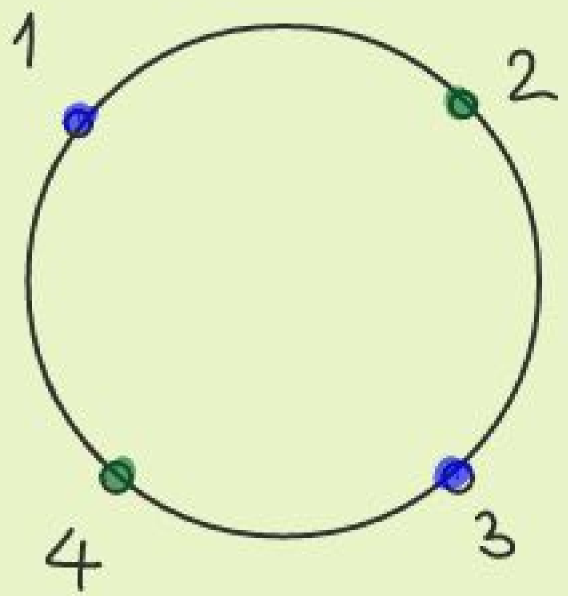
Let $\pi = (12)(34)$

Let $c = grgr$

Consider the case when $n = 4$ and $m = 2$




$S = \{1,2,3,4\}$

$C = \{\text{set of all colorings}\}$

$= \{gggg, gbgg, \ldots, \ldots\}$

Let $\pi = (12)(34)$

Let $c = gbgb$

then $\pi(c) = bgbg$

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How many such sets?

In the above problem the set of symmetries
— formed a cyclic subgroup.

In general:

$S$ — set of $n$ bead positions
$G$ — symmetries of $n$-gon.
$C$ — collection of $m^n$ colorings.

Count # diff. necklaces. [i.e. $G$ defines notion of equivalence.]

## Some definitions:

Given $\pi \in G$ let $C_\pi = \{c \in C \mid \pi(c) = c\}$

Given $c \in C$ let $G_c = \{\pi \in G \mid \pi(c) = c\}$

Note: $G_c$ is a subgroup of $G$.

Given $c \in C$ let $\overline{c} = \{\pi(c) \mid \pi \in G\}$

$\overline{c}$ is called the orbit of $c$.

Examples :

Suppose a group $G$ acts on a set of colorings $C$. For any coloring $c \in C$ we have

$$|G_c| \cdot |\bar{c}| = |G|.$$

<u>proof</u>:

The no. of equivalence classes of the set C in the presence of symmetries G is given by

$$N = \frac{1}{|G|} \sum_{\pi \in G} |C_\pi|$$

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