(CS 207) Discrete Structures 19 November, 2013

EndSem

- If you plan to use the results proved in class or from high-school math, please mention them clearly before using them.
- Please provide valid justifications/proofs for **all** questions.
- Make reasonable assumptions wherever necessary.
- Please be concise. Avoid writing irrelevant facts and results.
- Best of luck!

(Total marks: 50)

1. A permutation π on n elements is called an involution if π composed with itself gives the identity permutation. Let T_n denote the number of involutions on n elements. Give a recurrence for T_n. (Justify your answer.) (5)

2. If the edges of the complete graph on 17 vertices are colored with 3 colors then for any such coloring there exists a monochromatic triangle (a triangle with all edges having the same color). (5)

3. Prove or disprove the following for a connected graph G on n vertices: G has exactly one cycle if and only if it has exactly n edges. (5)

4. The city of Mumbai has n women and n men. The people are partitioned into two sets – those who live in south Mumbai and those who live in north Mumbai. There are k women from south Mumbai (i.e. $n - k$ from north Mumbai) and k men from south Mumbai (i.e. $n - k$ from north Mumbai). Every person would rather marry a person from south Mumbai than any person from north Mumbai. That is, in each preference list of a woman, the rank of any man from south Mumbai is higher than the rank of any man from north Mumbai. Similarly, in each preference list of a man, the rank of any woman from south Mumbai is higher than the rank of any woman from north Mumbai. Prove that in every stable matching every man from south Mumbai is married to a woman from south Mumbai. (5)

5. Flipkart has mn T-shirts. Each T-shirt is uniquely labelled by a pair (color, size)= (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq m$. (No two T-shirts are labelled with the same pair.) On the Flipkart webpage, all the T-shirts are displayed in an $n \times m$ array. Prove that for any such arrangement, there is a way to pick one T-shirt from each column such that the chosen set of T-shirts have different sizes. (8)

6. Burnside Lemma. (3+1+3+3)

Let the faces of a cube be numbered from 1 to 6. The symmetries of a cube can be represented by permutations of the 6 faces.

   (a) Let G denote the group of symmetries of the cube. Recall that any permutation can be written as a collection of cycles. For example, let π be the following permutation: $\pi(1) = 2, \pi(2) = 1, \pi(3) = 3$. Then π can be written as $\pi = (1, 2)(3)$. Let G_i denote the permutations in G which can be written as a collection of i cycles. Write down sizes of G_1, G_2, G_3, G_4, G_5 and G_6.

   (b) From part (a), compute the size of G.

   (c) Let C denote the set of all 2-colorings of the faces of the cube. $|C| = 2^6$. Using part (a) above, compute $\sum_{\pi \in G} |C_\pi|$, where $C_\pi := \{c \in C \mid \pi(c) = c\}$. Recall, $\pi(c)$ stands for the new coloring obtained by applying the symmetry π to the color vector $c \in C$.

   (d) Using Burnside Lemma count the number of distinct 2-colorings of the faces of the cube. Any two colorings are called *distinct* if one cannot be obtained from the other using symmetries from G.

7. Let $A_1, A_2, \ldots, A_n$ be subsets of $[n]$, where $[n]$ denotes $\{1, 2, \ldots, n\}$. Prove the following: (2+5)

   (a) $|\cup_{i=1}^{n} A_i| \leq \sum_{i=1}^{n} |A_i|$

   (b) $|\cup_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} |A_i| - \sum_{i,j \in [n], i<j} |A_i \cap A_j|$

8. Let $2^{\mathbb{N}}$ denote the set of all subsets of natural numbers. Let $(2^{\mathbb{N}}, \subseteq)$ denote the relation $\{(A, B) \mid A, B \in 2^{\mathbb{N}} \text{ and } A \subseteq B\}$. Observe that this is a partial order. Prove that it has an anti-chain with uncountably many elements. (5)