(CS 207) Discrete Structures 11 Sep, 2013

Mid-Sem

1. Partial orders and lattices (5+5+5)
An element $x \in S$ is said to be a *maximal* element of a partial order $(S, \preceq)$ if for any $y \in S$, if $x \preceq y$ then $x = y$. An element $z \in S$ is called a *lower bound* for $x, y \in S$ if $z \preceq x$ and $z \preceq y$. It is called a *greatest lower bound* if it is maximal in the set of lower bounds (with respect to $\preceq$). *Upper bound* and *least upper bound* can be defined similarly. A partial order is called a *lattice* if every pair of elements has a unique least upper bound and a unique greatest lower bound.

   (a) Prove that any finite non-empty partial order has a maximal element.

   (b) Give an example of a lattice.

   (c) Let us define a relation by the following set

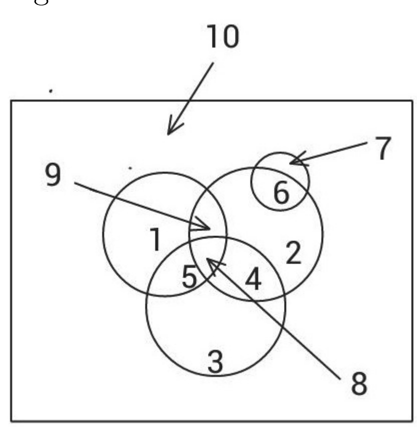
$$\{(x, y) \mid x, y \in \mathbb{R}^n \text{ and } \forall i \in [n], x_i \leq y_i\}$$

   Which of the following is it: Equivalence relation, partial order, lattice? (Justify your answer.)

2. Prove the following: (5+10)

   (a) Let α be such that $\alpha \in \mathbb{R}$ and $\alpha + 1/\alpha \in \mathbb{Z}$, then for every $n \in \mathbb{N}$, $\alpha^n + 1/\alpha^n$ is in $\mathbb{Z}$.

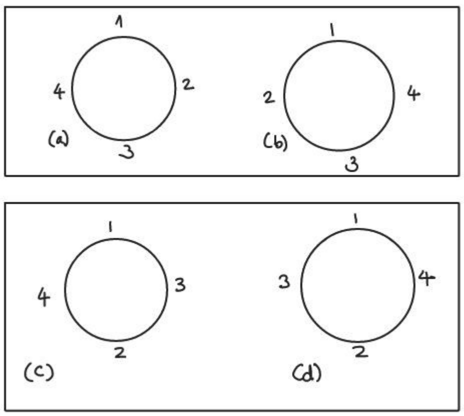
   (b) Let n circles be given in a plane. They divide the plane into parts. Any two parts are called neighbors if they share a common boundary. See for example the figure below.



In the figure above, $n = 4$. The given 4 circles divide the plane into 10 parts. To give an example of neighbors, the neighbors of 3 are $4, 5$ and 10, as each shares a common boundary with 3. However, 8 or 1 are not neighbors of 3 as they do not share any boundary with 3. Prove that (for any n) there is a way to color the parts with two colors such that no two neighbors get the same color.

3. Compute the following: (5+10)

   (a) Suppose n persons are sitting around a circular table. What fraction of the arrangements are distinct, i.e. do not have the same neighboring relation? For example in the figure below, (a) is the same as (b) and (c) is the same as (d), but arrangements (a) and (c) are distinct.



   (b) In how many ways can we select two (non-empty) disjoint sets from a set of size n? Suppose $S = \{1, 2, 3\}$, i.e. $n = 3$. We can select two non-empty sets as follows: $(\{1\}, \{2\})$, $(\{3\}, \{2\})$, $(\{1\}, \{3\})$, $(\{1, 2\}, \{3\})$, $(\{2, 3\}, \{1\})$, $(\{1, 3\}, \{2\})$ i.e. there are 6 ways. Note here that $(\{2\}, \{1\})$ is considered to be the same as $(\{1\}, \{2\})$.

4. For any positive integer n, let $\varphi(n)$ denote the number of positive integers $\leq n$ which are coprime to n. For example, $\varphi(2) = 1, \varphi(3) = 2$. We will define $\varphi(1) = 1$. Let T_n denote the set of all prime divisors of n. Prove using inclusion and exclusion that (15)

$$\varphi(n) = n \prod_{p \in T_n} \left(1 - \frac{1}{p}\right)$$