Quiz 1

1. Prove or disprove the following: 10

   (a) There is an injective map from $\mathbb{Z}$ to $\mathbb{N}$

   (b) Let $R = (S, \preceq)$ be a relation. Let R^2 be defined as $\{(a, b) \mid a, b \in S, \exists c \in S \text{ s.t. } (a, c) \in R, (c, b) \in R\}$. If R is symmetric then so is R^2.

   (c) $\sqrt{3}$ is irrational.

   (d) If $T(n) = T(\lfloor n/2 \rfloor) + n$ and $T(0) = 0$ then $T(n) \leq 3n$.

   (e) The set of irrational real numbers is countable.

2. Show that: 2+4

   (a) there is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

   (b) there is a bijection from $\mathbb{Q}$ to $\mathbb{N}$.

3. Prove that if R is a partial order then $R^2 = R$. 4

4. Prove the following: 2+3

   (a) How many relations on the set $\{1, 2, \ldots, n\}$ are simultaneously reflexive and symmetric?

   (b) How many relations on the set $\{1, 2, \ldots, n\}$ are simultaneously reflexive and anti-symmetric?

5. Prove or disprove: there is a bijection from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. 5

Quiz 1

1. Give an example of a non-abelian group with a non-trivial abelian subgroup. A subgroup H of a group G is called non-trivial if it not the entire group G and is not the identity. (Give both the group and its subgroup.) (1)

2. Prove that every simple undirected graph on n vertices, where $n \geq 2$, has at least two vertices with the same degree. (1)

3. Stable marriage (1+1+1)

   (a) Give an instance of the stable marriage problem (i.e. give a bipartite graph $G = (X, Y, E)$ and for every $x \in X$ give an ordering of x for every vertex $y \in Y$) such that it has exactly two possible stable matchings.

   (b) Consider the following graph: $G = (X, Y, E)$ such that $|X| = |Y| = 3$. $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. And let X denote the set of boys (i.e. they propose as per the Gale-Shapley algorithm). Give the output of Gale-Shapley for the following ordering:

   $1 : a > b > c \quad a : 1 > 3 > 2$
   $2 : a > b > c \quad b : 1 > 3 > 2$
   $3 : a > b > c \quad c : 1 > 3 > 2$

   (c) How many stable matchings does the above structure have? List them.

4. If at least $n+1$ integers from $\{1, 2, \ldots, 2n\}$ are selected, then some two of the selected integers are mutually prime. Two numbers are said to be mutually prime if their greatest common divisor is 1. (2)

5. How many nonnegative integer solutions do the following equations have? (Justify your answer with a complete proof.) (1+1)

   $(i) \sum_{i=1}^{k} x_i = n$ $(ii) \sum_{i=1}^{k} x_i \leq n$

6. Give an example of an infinite group whose every non-trivial subgroup is also infinite. Refer to Question 1 for the definition of a non-trivial subgroup. (Justify your answer.) (1)

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Quiz 4

Roll no: Total marks: (35)

1. Let $G = (S_3, *)$ denote the group in which S_3 denotes the set of all permutations on three elements and $*$ denotes the composition of permutations. Write all the (non-trivial) subgroups of G. (2)

2. You are in-charge of the IITB guesthouse. The guesthouse has 100 rooms. You decide to follow the following strategy to accommodate the guest requests: as soon as a request appears, you uniformly and randomly pick a number between 1 to 100 and if the room numbered with this chosen number is free then you allot the room to the guest, else you declare that there is no vacancy. Suppose you get 5 requests one after the other: (2+2)

   (a) What is the probability that you allot rooms to all 5 guests?

   (b) What is the probability that you allot a room to exactly one guest?

3. Give a graph on $2n$ vertices such that it has exactly two perfect matchings. (2)

4. How many integer sequences $(a_1, a_2, \ldots, a_{n-1})$ are there such that $1 \leq a_1 \leq a_2 \ldots \leq a_{n-1}$ and $a_i \leq i$ for every i? Give a closed form in terms of n. (3)
Hint: This is a quantity equal to something we saw in class.

5. In how many ways can 5 horses go through the finish for the following two cases? (1+2)

   (a) assuming there are no ties?

   (b) assuming there are ties?

6. How many words of length 5 over the alphabet $\{0, 1, 2, \ldots, 9\}$ are there with the following properties? (1+1+1+1)

   (a) Strictly increasing digits.

   (b) Strictly increasing or deceasing digits.

(c) Increasing digits.

(d) Increasing or deceasing digits.

7. Let $x_0 = 2, x_1 = 7$ and $x_{n+1} = 7x_n - 12x_{n-1}$. Find a closed expression for x_n. (2)

8. Let t_n denote the number of ways of tiling a $2 \times n$ grid with a 2×1 tile. Give a recurrence for t_n. (1)

9. Let t_n denote the number of ways of tiling a $2 \times n$ grid with a 2×1 or with a 2×2 tile. Give a recurrence for t_n. (2)

10. Consider a unit cube. It has 6 faces. A rigid motion of the cube is called a symmetry if it maps the object to itself (preserving colors of the faces). (1+1)

    (a) Suppose the top and bottom faces are colored black and all other faces are white. How many symmetries does this cube have?

    (b) Suppose the top and the front faces are colored black and all other faces are white. How many symmetries does this cube have?

11. Give a tournament on 4 vertices in which every vertex is a king. (2)

12. Find mistakes in the following proofs. (Please specify *all* the mistakes clearly.) (2+2)

Lemma If n is an even number and $n \geq 2$ then n is a power of two.

Proof: **Base case:** $n = 2$ is the first power of 2.
Induction step: Assume that k is a number greater than 2. If k is odd then there is nothing to prove. Else $k = 2\ell$ for some $2 \leq \ell \leq k$. By induction hypothesis there exists an index i such that $\ell = 2^i$ and therefore $k = 2^{i+1}$.

Lemma All n node rooted binary trees have height $n - 1$, where the height of the tree equals the length of longest root to leaf path.

Proof: **Base case:** For $n = 2$, it is trivially true.
Induction step: Suppose T_n is a tree on n nodes. Obtain a tree on $n + 1$ nodes, say T_{n+1}, by attaching the $n+1^{th}$ node to one of the leaves of T_n. Note that if T_n is a binary tree then T_{n+1} is a binary tree. The height of T_{n+1} is one more than the height of T_n. By induction hypothesis height of T_n equals $n - 1$. Putting these things together proves

the statement.

13. The numbers from 1 to 10 are written on a board. Akbar and Birbal take alternate turns to play the game (Akbar plays first), each putting either a $+$ or a $-$ sign in front of any number which has no sign so far. Therefore, the game goes on for 8 steps at the end of which each number has a sign. At the end of the game Akbar rewards Birbal with as many gold coins as the absolute value of the sum of the signed numbers. Akbar wants to give as few gold coins to Birbal as possible and Birbal wants to earn as many gold coins as possible. (2+2)

    (a) What is the maximum number of coins Birbal can win?

    (b) If the numbers were from 1 to $2n$ then what is the maximum coins that Birbal can win?