Tutorial 1

1. Prove the following properties of Fibonacci numbers using induction, where Fibonacci numbers are defined as follows:
   $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

   (a) Prove that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$

   (b) Prove that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$

   (c) Prove that $F_1+F_3+F_5+\ldots+F_{2n+1} = F_{2n+2}$, and $1+F_2+F_4+F_6+\ldots+F_{2n} = F_{2n+1}$

2. Using the well-ordering principle prove the following statement: Any $n \in \mathbb{Z}^+$, if $n > 1$ then it can be factored as a product of primes.

3. Find bugs in the following proof:

   Claim 0.1 (bogus)**.** *Every Fibonacci number is even.*

   Faulty proof. The proof is by using the well-ordering principle (WOP). Suppose there are odd Fibonacci numbers. Let C be the set of all such numbers. By the WOP, we know that there is a minimum number m such that F_m is odd. As F_0 is even, $m > 0$. Since m is the minimum in C, we know that F_{m-1}, F_{m-2} are both even. But by the definition of $F_m = F_{m-1} + F_{m-2}$, F_m is a sum of two even numbers. Therefore, it must be even. Therefore, our assumption that there are odd Fibonacci numbers must be wrong, thereby proving the claim! $\square$

4. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show (using induction) that there is a car which can complete a lap by collecting gas from the other cars on its way around.

5. There is a building with n floors. You are given two identical eggs. There is some floor $h \leq n$ such that if the eggs are thrown from h then they break and if the eggs are thrown from another floor $k < h$ then they do not break. Give an optimal strategy to compute h.

Tutorial 2

1. Prove or disprove the following:
   (a) If A, B are countable then so is $A \cup B$
   (b) Let A, B be two nonempty sets. If there is a bijection from A to B then there is a bijection from $A \times A$ to $B \times B$.
2. Give a bijection from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$
3. Prove that if A is countably infinite and B is a finite set then $A \cup B$ is countably infinite.
4. Give a bijection from $\mathbb{R}$ to set of all subsets of $\mathbb{N}$.
5. **Akbar and Birbal**

   Here is a slightly modified tale of the king Akbar and his brilliant council member Birbal. Akbar and Birbal decided to play a game. Akbar decided that he will be the one who will think of a natural number and Birbal had to guess the number. Birbal could only guess one number every day. If Birbal guessed correctly, the king promised him 1000 gold coins. If the number is wrong, the game continues. Birbal was said to lose, if the game went on forever.

   (a) What is Birbal's strategy to win?
   (b) If Akbar's number, say $x \in \mathbb{Z}$, then does Birbal have a winning strategy? If so, what is it? (If not – justify).
   (c) Suppose the king chooses an ordered pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ and Birbal is suppose to guess the pair. The guess is correct only if Birbal gets both the numbers right and in that order. Does Birbal have a winning strategy? If so, what is it? (If not – justify).
   (d) Suppose Akbar chooses an arbitrary but finite set $A \subset \mathbb{N}$ of cardinality at most n and Birbal knows n and is suppose to guess a set every time. The guess is correct only if Birbal guesses the exact set A. Does Birbal have a winning strategy? If so, what is it? (If not – justify).
6. Induction

   Let $x_0 = 1$ and $x_{i+1} = \left(\sqrt{2}\right)^{x_i}$. Prove that

   (a) $\forall i \in \mathbb{N}$, $x_{i+1} > x_i$
   (b) $\forall i \in \mathbb{N}$, $x_i < 2$

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Tutorial 3

1. **A part of the proof of the Schröder Bernstein theorem**

   Recall the following things defined in class. Let A, B be two sets and let g be an injective map from A to B and h be an injective map from B to A. Let Range(g) be called B_0 and let $B_1 = B \setminus B_0$. Let an element $b \in B$ be called h-good if there exists an element $\beta \in B_1$ and there exists an $n \in \mathbb{N}$ such that $b = (g \odot h)^n \beta$. Also recall the function $f : A \rightarrow B$ defined in class as follows:

   $f(a) = \begin{cases} h^{-1}(a) & \text{if } g(a) \text{ is } h\text{-good} \\ g(a) & \text{otherwise} \end{cases}$

   Prove that f is surjective.

2. Prove that Induction implies strong induction.

3. Prove that strong induction implies the well-ordering principle.

4. Let $X = \{1, 2, \ldots, n\}$ and consider the partial order $(\mathcal{P}(X) \subseteq)$. Prove the following:

   (a) A chain is maximal if no other element can be added to it. There are at most $n!$ maximal chains.

   (b) Let $\mathcal{M}$ be any anti-chain. Let $\mathcal{P} = \{(C, L) \mid C \text{ is a maximal chain, } L \in C \cap \mathcal{M}\}$. Using (a), prove that $|\mathcal{P}| \leq n!$

5. **Partial order**

   Let (S, R) denote a partially ordered set S under the order R. An antichain is the subset $S' \subseteq S$ of S such that $\forall x, y \in S'$, $x \neq y$, $x \not R y$.

   (a) Give an example of an antichain.

   (b) What is the length of a maximal antichain in $(\mathbb{N}, \leq)$, $(\mathcal{P}_A, \subseteq)$. Here $A = \{1, 2, \ldots, 10\}$ and $\mathcal{P}_A$ denotes the power set of A.

   (c) Show that a chain and an antichain intersect in at most 1 point.

6. **Relational data of BTech 2**

   Consider the database of BTech2 consisting of three parameters: roll numbers, JEE rank, and CPI. Let x and y be two students. We define two relation $\succ, >>$ as follows: $x \succ y$ iff $CPI(x) - CPI(y) > 1.5$ and $x >> y$ iff $JEE(x) \leq JEE(y)$ and $CPI(x) \geq CPI(y)$.

   (a) Which among the following properties are satisfied by $\succ$ and $>>$: reflexive, transitive, symmetric, antisymmetric.

   (b) A relation is called an equivalence relation if it is reflexive, transitive, and symmetric. Define a relation on this database which is an equivalence relation.

   (c) Let $\sim$ be an equivalence relation. Let $C_x = \{y \mid x \sim y\}$. The set C_x is called an equivalence class of x. How many equivalence classes are there in the relation that you defined in the previous subpart?

7. How many functions exist from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$? How many injective functions exist from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$?

8. Each bead on a necklace with three beads is colored either black or white. Necklaces N_1, N_2 are said to be related if N_2 is N_1 or can be obtained from N_1 by flipping around the center of N_1. Is it an equivalence relation? If not, then which property does it not satisfy? If it is, then what are the equivalence classes?

Tutorial 4

1. Give a recurrence for the number of distinct ways of writing a positive integer n as a sequence of 1s and 2s such that the sum of the elements of the sequence is exactly n? For example, for $n = 3$, (1,2), (2,1), (1,1,1) are the three distinct sequences that add to 3.

2. Let $S = \{0, 1, 2\}$. An n-word over the set S is a string of length n, say $x_1 x_2 \ldots x_n$ such that for each $1 \leq i \leq n$, $x_i \in S$. How many n words are there over S such that neighbors differ by at most 1. First give a recurrence and then solve the recurrence.

3. Solve the following recurrences:
   (a) $F(n) = 3F(n-2) - 2F(n-3)$
   (b) $t(n) = 2\sqrt{n} \cdot t(\sqrt{n}) + n$
   (c) $t(n) = \sqrt{n} \cdot t(\sqrt{n}) + n$

4. A matrix with entries from $\{0, 1\}$ is called a permutation matrix if every row and every column has exactly one 1-entry. Count the number of distinct such $n \times n$ permutation matrices.

5. Let $D(n)$ denote the number of derangements. Prove that for $n \geq 1$ $D(n) = nD(n-1) + (-1)^n$

6. Find coefficients of
   (a) x^5 in $(1 + x)^{10}$
   (b) $x^3 y^8$ in $(3x + 2y)^{10}$

7. Use generating functions to prove the following: $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ for $n \in \mathbb{N}$

8. Using ideas used for estimating $n!$, approximate $\sum_{i=1}^{n} \sqrt{i}$
   Prove that $\frac{2}{3}(n^{3/2} - 1) + 1 \leq \sum_{i=1}^{n} \sqrt{i} \leq \frac{2}{3}(n^{3/2} - 1) + \sqrt{n}$

9. Give a bijection between A and B, where A is all ways of selecting n burfees from k different types of burfees and B is all $n + k - 1$ bits strings over $\{0, 1\}$ with exactly $k - 1$ 1s

10. We will say that a Ruppee note is invalid if any even number appears more than once. If every Ruppee note has k digits from $\{0, 1, 2, \ldots, 9\}$ then what fraction of the bills are valid?

11. How many 5 digit numbers have exactly one 8 in them and the sum of their digits add to 19? (The digits come from $\{0, 1, 2, \ldots, 9\}$

12. (*) A permutation $x_1 x_2 \ldots x_{2n}$ is called *good* if for at least one $i \in \{1, 2, \ldots, 2n-1\}$, $|x_{i+1} - x_i| = n$. Let $G(n)$ denote the number of good permutations on $2n$ elements. Give a recurrence for $G(n)$. Prove that for $n \geq 5$, $G(n) \geq n!/2$.

Tutorial 5

- Solve all questions. Discuss solutions with TAs during TA meeting hours.

1. Find the number of distinct ways of writing a positive integer n as a sequence of 1s and 2s such that the sum of the elements of the sequence is exactly n? For example, for $n = 3$, (1,2), (2,1), (1,1,1) are the three distinct sequences that add to 3.

2. Solve the following recurrences:

   (a) $F(n) = 3F(n-2) - 2F(n-3)$

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3. A matrix with entries from $\{0, 1\}$ is called a permutation matrix if every row and every column has exactly one 1-entry. Count the number of distinct such $n \times n$ permutation matrices.

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5. Use generating functions to prove the following: $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ for $n \in \mathbb{N}$

Tutorial 6

1. Show that any finite graph contains at least two vertices lying on the same number of edges.
2. Let $a_1, a_2, \ldots, a_n$ be n (not necessarily distinct) integers. There exists a subset of these numbers such that the sum of number in the subset is divisible by n.
3. Given five points in the plane, no three of them colinear, some four of them form a convex quadrilateral.
4. Given five points in the plane, no three of them colinear, the number of convex quadrilaterals formed is odd.
5. There are n rows of students and in each row there are n students. Let us call this initial arrangement A_0. A teacher sorts each column by the increasing order of students' heights (say from front to back) to obtain another arrangement, say A_1. She then sorts each rowof A_1 by increasing order of students' heights (say from left to right) to get a new arrangement A_2. Show that in A_2 columns are still in the sorted order.
6. A plane is colored with two colors. Prove that there exists a rectangle with all its corners colored with the same color.
7. A plane is colored with three colors. Prove that there exist two points at distance 1 both with the same color.

Tutorial 7

Solve the following problems from Douglas West:

1. 1.2.17, 1.2.20, 1.2.38, 1.2.43, 1.3.20, 1.3.31, 1.3.41
2. 3.1.8, 3.1.11, 3.1.18, 3.1.24, 3.1.28