Depth 4 lower bounds for elementary symmetric polynomials

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Joint work with
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Homogeneous vs. inhomogeneous

Degree of all the input polynomials to any $\sum$ gate is the same.

Known results

Small inhomogeneous formulas exist

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- Conclude that $s \geq \mathcal{L}/\mathcal{U}$

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$$M[(R_1, \ldots, R_\tau), (S_1, \ldots, S_\tau, T)] = 1 \quad \text{if} \begin{cases} T \subseteq R_1 \\ S_1 \subseteq R_1 \cup R_2 \\ S_2 \subseteq R_2 \cup R_3 \\ \vdots \\ S_{\tau-1} \subseteq R_{\tau-1} \cup R_\tau \\ S_\tau \subseteq R_\tau \end{cases}$$

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$$M[(R_1, \ldots, R_\tau), (S_1, \ldots, S_\tau, T)] = 1 \quad \text{if} \begin{cases} T \subseteq R_1 \\ S_1 \subseteq R_1 \cup R_2 \\ S_2 \subseteq R_2 \cup R_3 \\ \vdots \\ S_{\tau-1} \subseteq R_{\tau-1} \cup R_\tau \\ S_\tau \subseteq R_\tau \end{cases}$$

Lemma (Main technical lemma)

$Rank(M) = min\{\#cols, \#rows\}(1 - o(1))$

Open problems

Some problems arising from the work

For all $D \in [n]$, any depth 4 homogeneous $\Sigma\Pi\Sigma\Pi^{[t]}$ formula computing S_n^D requires size $n^{\Omega(D/t)}$.

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Give an explicit $f(X) \in \mathbb{F}[X]$ on n variables such that depth 3 inhomogeneous formula computing f require size $n^{\omega(1)}$?

Open problems

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Prove that the above matrix has rank $min\{\#cols, \#rows\}(1 - o(1))$

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Thank You!