Isolation Lemma for Directed Reachability and NL vs. L

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Isolation lemma for NP

SAT

Given a Boolean formula ϕ determine whether ϕ has a satisfying assignment.

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Valiant Vazirani Isolation Lemma

There is a randomized polynomial time isolation procedure for SAT with success probability $\Omega(\frac{1}{n})$

Bumping up the success probability

Determinizing Isolation Procedure for SAT

Can one get rid of the randomness in the Isolation Procedure for SAT?

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The success probability of the Isolation Procedure for SAT can be made greater than $2/3$ if and only if $\mathrm{NP} \subseteq \mathrm{P/poly}$.

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[DKvMW] *Is Valiant Vazinai's isolation probability improvable? Dell, Kabanets, van Melkebeek, Watanabe, Computational Complexity, 2013.*

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Output: yes if and only if there is a directed path from s to t in G.

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Success probability of the Isolation Procedure for **Reach**

Theorem (Main result)

There exists a randomized isolation procedure for **Reach** *with success probability greater than* $2/3$ *if and only if* $NL \subseteq L/poly$.

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There exists a randomized isolation procedure for a hard problem in LogCFL that runs in L/poly with success probability greater than 2/3 if and only if LogCFL $\subseteq$ L/poly.

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There exists a randomized isolation procedure for SAT that runs in L/poly with success probability greater than $2/3$ if and only if NP $\subseteq$ L/poly.

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Definition (**PrUReach**)

Given: $G \in \mathsf{Yes}_{\textbf{Reach}} \cup \mathsf{No}_{\textbf{Reach}}$

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Given: $G \in \mathrm{Yes}_{\mathbf{Reach}} \cup \mathrm{No}_{\mathbf{Reach}}$

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Step 1 Prove that $\mathsf{NL} \subseteq \mathsf{L}/\mathsf{poly}$ if and only if **PrUReach** $\in \mathsf{L}/\mathsf{poly}$.

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Definition (Min-unique graph [GW, RA])

A weighted directed graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{N}$ is said to be *min-unique* if between every pair of vertices the minimum weight path is unique.

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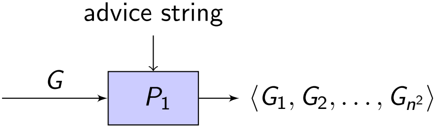

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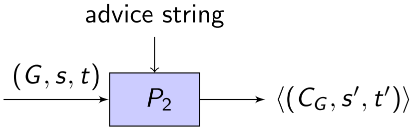

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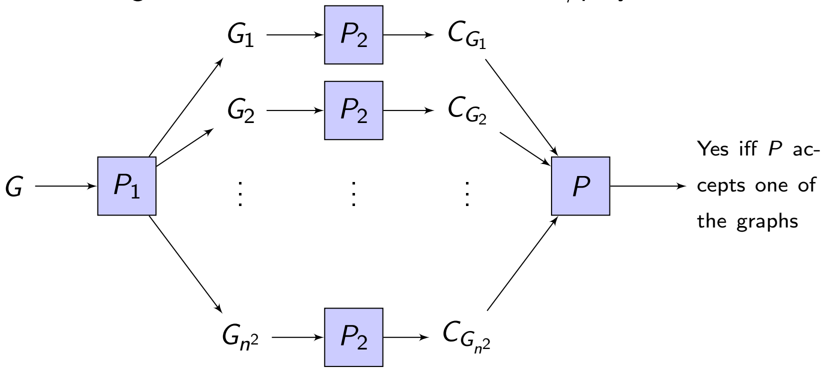

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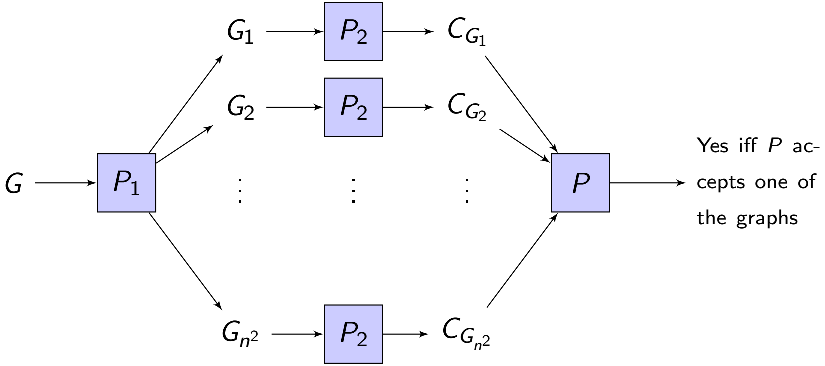

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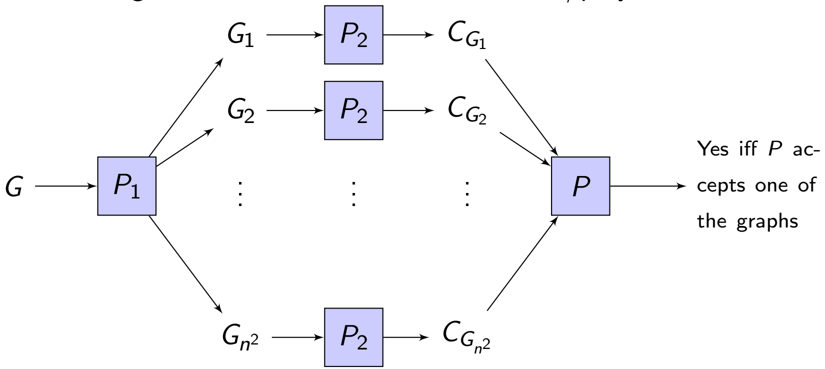

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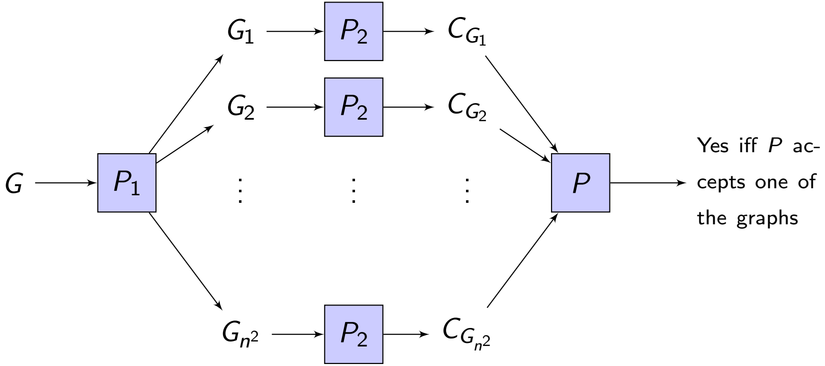

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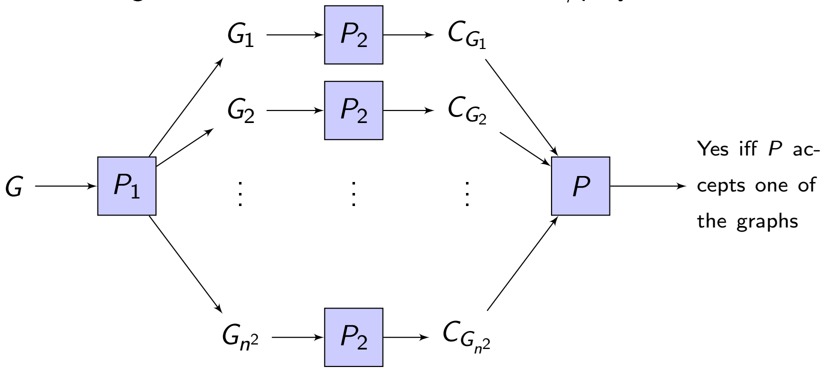

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C_G can be computed in L.

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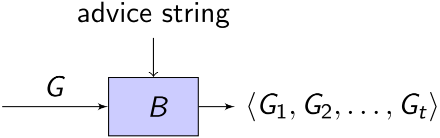

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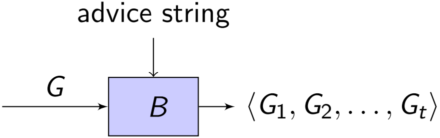

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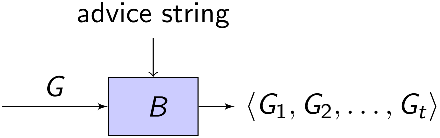

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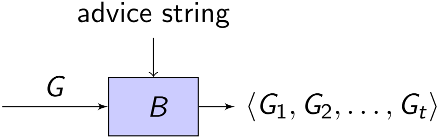

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Let $B(G + H) = \langle G_1, G_2, \ldots, G_t \rangle$.

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Given H, π as advice and G as input, whether (G, H) is good or not can be decided in L/poly.

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Design the advice strings

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Putting it together

Overall, this gives a L/poly algorithm for **PrUReach**.

Thank You!