A Near-Optimal Depth-Hierarchy Theorem for Small-Depth Multilinear Circuits

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Joint work with

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More Resources

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More Resources ⇒ More power?

Power of resources

Turing machines

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Non-explicit separations.

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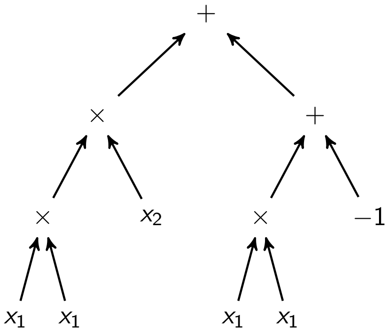
Today we will focus on arithmetic formulas as a model of computation.

A model of computation for polynomials

Arithmetic formula



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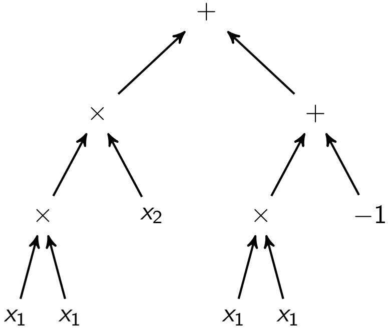

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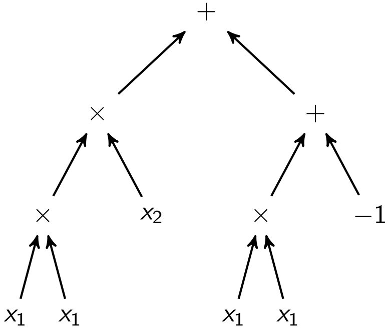

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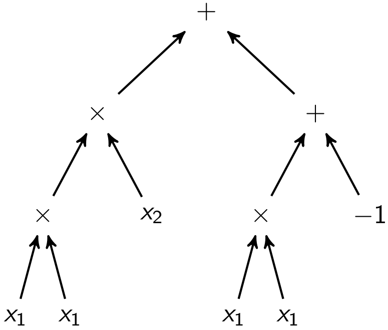

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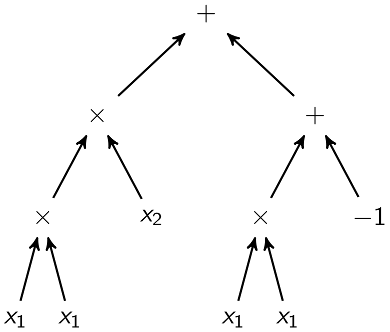

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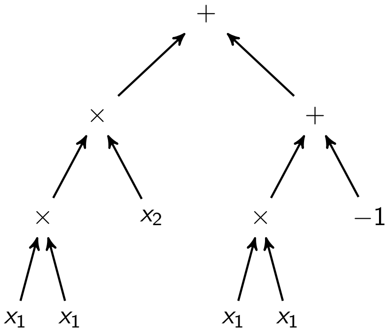

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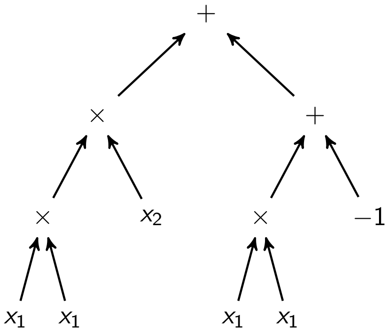

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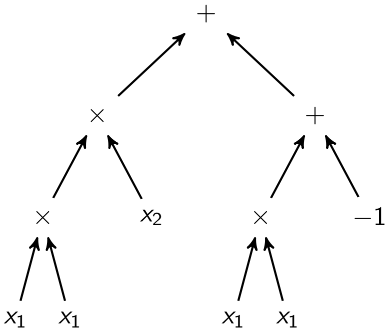

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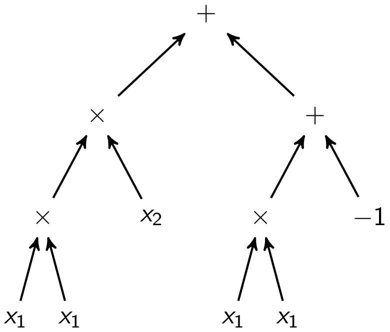

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The maximum number of product gates along any leaf to root path is its product-depth. (Closely related to the depth.)

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We will focus on small product-depth formulas.

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It may not always be a succinct representation.

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This is a $\prod \sum$ formula for the same polynomial.

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- Extended and appended by a line of work. [Raz06,RSY07,RY09,DMPY12,KV17].

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This $\sum\prod\sum$ realization is non-multilinear!

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[Kayal, Nair, Saha, 15] show that this is not possible.

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[Kayal, Nair, Saha, 15] show that this exponential blow-up is essential while going from $\Delta = 2$ to $\Delta = 1$.

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Is the blow-up essential?

More Resources

Depth hierarchy theorem

More Resources ⇒

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More Resources ⟹ More power?

More Product-depth

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Arithmetic circuit complexity world

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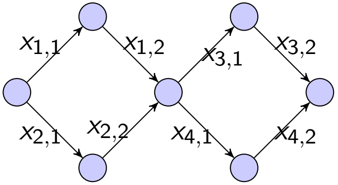

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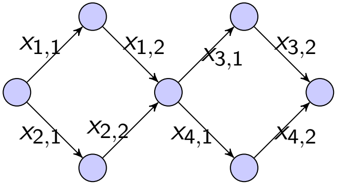

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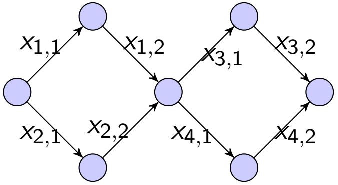

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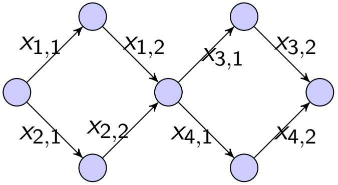

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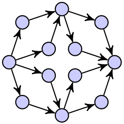

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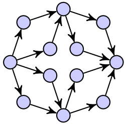

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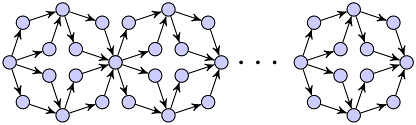

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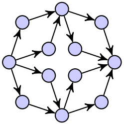

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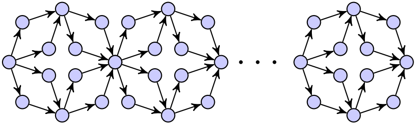

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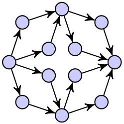

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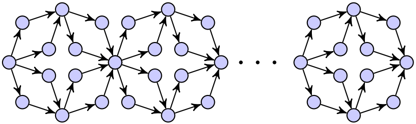

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$P^{(1)}$ is a polynomial over $n_1 := 8(2m)$ many variables and has product-depth 2, size $O(m) = O(n_1)$ formula.

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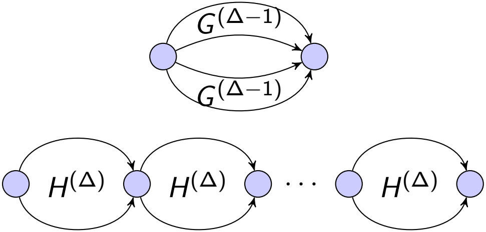

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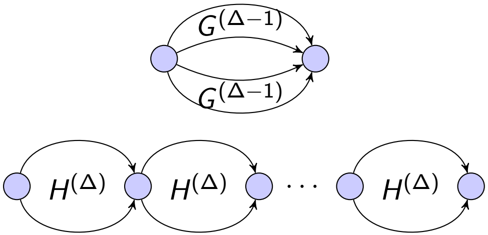

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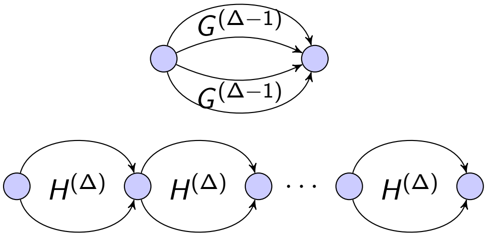

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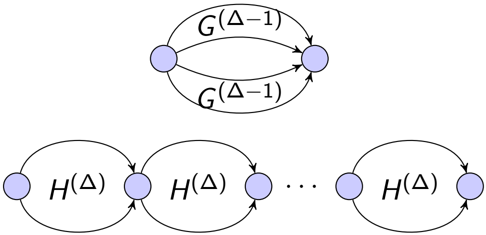

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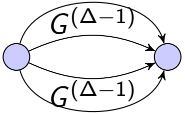

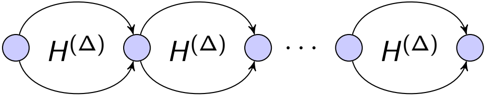

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Proof details

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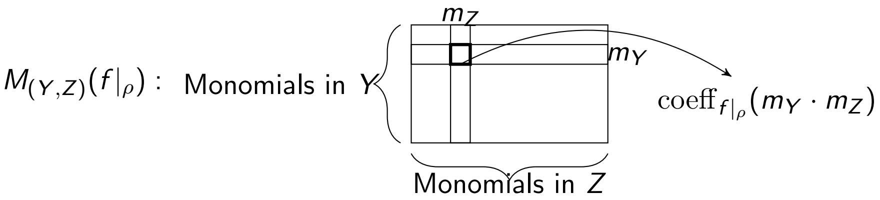

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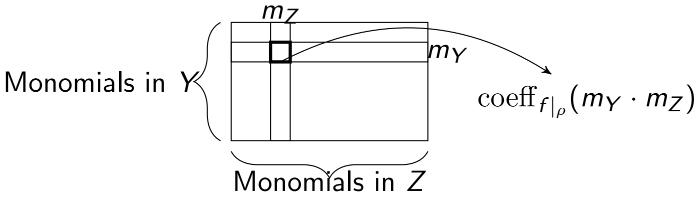

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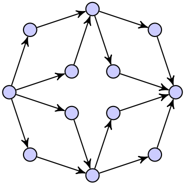
A random $\rho : X \to Y \cup Z \cup \mathbb{F}$

Map every copy of $H^{(1)}$ uniformly at random to one of the three possibilities.



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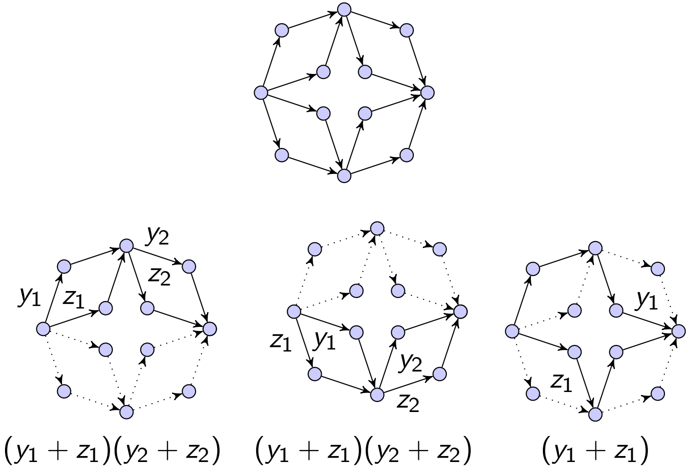
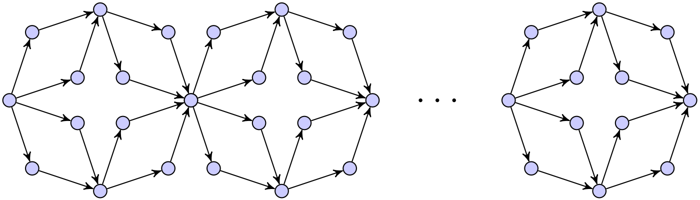


Figure: Map ρ applied to each copy of $H^{(1)}$. Edges that are not labelled have their variables set to 1. Dotted edges have their variables set to 0.

Random map ρ

Recall that $G^{(1)}$ is m copies of $H^{(1)}$.



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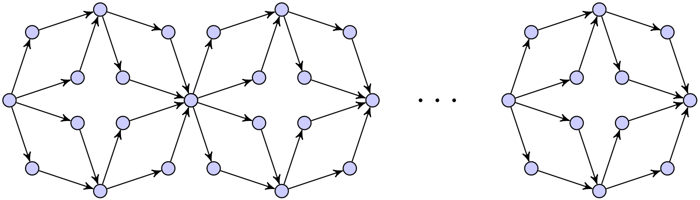

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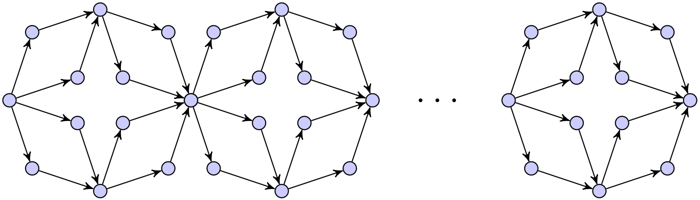

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A carefully chosen ρ at each *level* of the polynomial.

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