# A #SAT Algorithm for Small Constant-Depth Circuits with PTF gates.

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Joint work with

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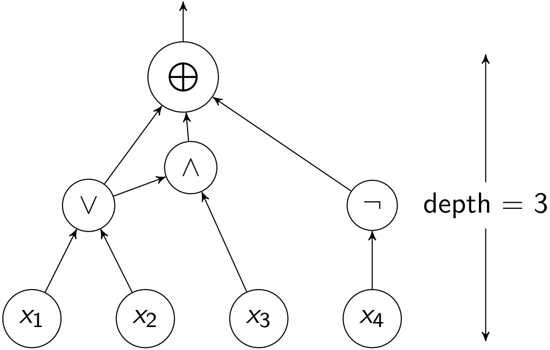


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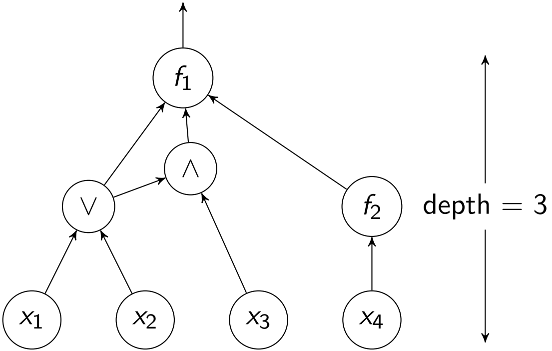


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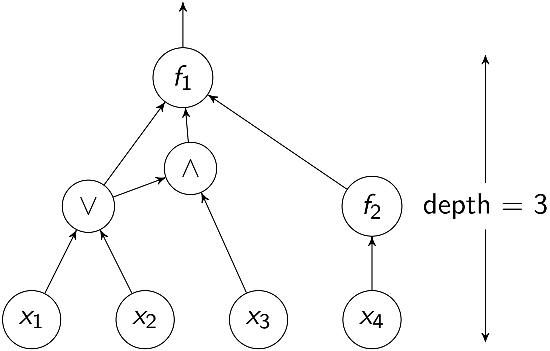


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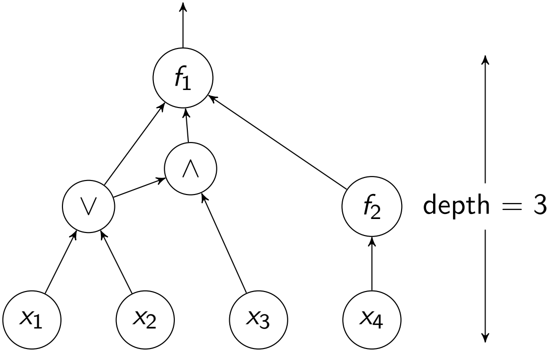


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Constant-depth circuits: $d$ is independent of $n$.

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Can we design an algorithm that takes time $2^n / n^{\omega(1)}$ when $|C|$ is small, say $\mathrm{poly}(n)$?

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This intuitive connection has been formalised to derive lowerbounds for various interesting classes of circuits.

[Paturi, Pudlák, Zane 1997],[Paturi, Pudlák, Saks, Zane 2005], [Williams 2010], [Williams 2011].

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Polynomial Threshold circuits are a natural generalization of $\mathsf{TC}^0$.

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The last two algorithms also work for #SAT.

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*Fix any constant $k$, there is a zero-error radomized algorithm that solves the #SAT problem for a single $k$-PTF in time $\mathrm{poly}(n, M) \cdot 2^{n-s}$, where $s = \tilde{\Omega}(n^{1/k+1})$.*

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Some comments on zero-error randomized algorithms.

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Step 2 On input $C \in \mathcal{C}$,
set variables $x_{m+1}, \ldots, x_n$ to all possible Boolean values.

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A 2-step procedure to solve satisfiability for class $\mathcal{C}$ of circuits.

Step 1 Use brute-force to solve all instances on $m$ inputs. Typically $m = n^{\varepsilon}$.

Store all answers (SAT or not SAT) for each in a table $\mathcal{T}$.

Takes time $\exp(m^{O(1)}) \ll 2^n$.

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If look-up can be done in $\mathrm{poly}(|C|)$ time, then this step takes time $O(2^{n-m} \cdot \mathrm{poly}(|C|)) \ll 2^n$.

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Expensive to compute

Even for LTFs computing Chow parameters is known to be NP-hard. [O'Donnell, Servedio 2011].

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- It computes a function $F : \mathbb{R}^r \to \{-1,1\}^{|H|}$ such that
- Given an input $\text{coeff}(P') \in \mathbb{R}^r$, $F(\text{coeff}(P'))$ is the truth table of the $k$-PTF defined by $P'$.
- The decision tree depth is $\Delta = O(r \log r \log |H|) = O(m^{k+1} \log m)$.

# Linear Decision Tree for $k$-PTF

## Linear Decision Trees [Kane, Lovett, Moran, Zhang 2017]

Given a $k$-PTF $P'$ on $m$.

Run the algorithm of [KLMZ 2017] with $r = \sum_{i=1}^{m} \binom{m}{i}$ and a set $H = \{e_b \mid b \in \{-1,1\}^m\} \subseteq \{-1,1\}^r$.

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The tree can be constructed in time $2^{O(\Delta)} = \exp(m^{k+1})$ time.

# Memoization for $k$-PTF

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# Small-depth $k$-PTF circuits

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Note that there are two crucial steps where [KL 2018] use the sparcity assumption.

Develop strategies that work for these two steps even when the $k$-PTF gates are not sparse.

# Satisfiability algorithms vs. SAT solving

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Some techniques in this talk could be of general interest!

# Conclusion and open problems

Better than brute-force algorithms for

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Open problems

- Can the techniques here give better-than brute-force algorithms in the subquadratic regime?
- Other connections from learning algorithms to satisfiability algorithms?
- Can this technique inspire any SAT solving heuristic?

Thank You!