Perspective Projection

Image Plane or Projection Plane

Projectors

Centre of Projection

Object

Parallel Projection

Image Plane or Projection Plane

Projectors

Centre of Projection?

Object

Parallel Projection

Orthographic Projection

• Multiviews (x=0, y=0 or z=0 or principal planes).
• True size or shape for lines.
• For projection on the z=0 plane we get the projection matrix as

\[
P(z=0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

Parallel Projection

Axonometric Projection

• Transform and then project using an orthographic projection such that at multiple adjacent faces are visible – better representation of a 3D object using 1 view. Face parallel to projection plane shows true shape and size.
• If \(U\) be the matrix formed by stacking up the unit vectors along the three axes, and \(T\) be the axonometric projection, then

\[
T.U = T = \begin{bmatrix} 1 & 0 & 0 & x' \\ 0 & 1 & 0 & y' \\ 0 & 0 & 1 & z' \end{bmatrix}
\]

• The foreshortening ratios for each projected principal axes are then given by:

\[
f_x = \sqrt{x'^2 + y'^2} \quad f_y = \sqrt{x'^2 + y'^2} \quad f_z = \sqrt{x'^2 + y'^2}
\]
Parallel Projection
Axonometric Projection
- If \( T \) be the matrix formed by stacking up the unit vectors along the three axes, and \( T \) be the axonometric projection, then

\[
T.U = T.
\]

- The foreshortening ratios for each projected principal axes are then given by:

\[
f_x = \sqrt{x_x^2 + y_x^2}, \quad f_y = \sqrt{x_y^2 + y_y^2}, \quad f_z = \sqrt{x_z^2 + y_z^2}.
\]

Depending on the kind of foreshortening they cause we can have three types of axonometric projections
- Trimetric (all foreshortenings are different)
- Dimetric (two foreshortenings are the same)
- Isometric (all foreshortenings are the same)

Assuming we rotate by \( \phi \) and \( \theta \) before we do the projection on the z=0 plane.

Now we apply this axonometric projection \( T \) to \( U \).
Parallel Projection

Axonometric Projection

- So the foreshortening ratios become

\[
\begin{align*}
    f_x^2 &= \cos^2 \phi + \sin^2 \theta \sin^2 \phi \\
    f_y^2 &= \cos^2 \theta \\
    f_z^2 &= \sin^2 \phi + \sin^2 \phi \cos^2 \phi
\end{align*}
\]

- For isometric projections, if we solve for \( f_x = f_y = f_z \), then we get \( \theta = 35.26^\circ \) and \( \phi = \pm 45^\circ \).

Parallel Projection

Oblique Projection

- The projectors are parallel to each other but they are not perpendicular to the plane of projection.
- Only planes parallel to plane of projection show true shape and size.

Perspective Projection

- Projectors converge at a finite centre of projection.
- Parallel lines converge.
- We get non-uniform foreshortening.
- Shape is not preserved.
- We see in perspective – so perspective viewing seems natural and helps in depth perception.

A digression into art

13\textsuperscript{th} century, Arezzo by Giotto
A digression into art

- Early 15th century, The Little Garden of Paradise

- 15th century, The Baptistry in Florence, Filippo Brunelleschi

- 15th century, Fresco of Holy Trinity, Masaccio

A digression into art

- 15th century, School of Athens, Raphael

Perspective Projection

\[
\begin{align*}
    & P(x, y, z) \\
    \Rightarrow & y' = \frac{y}{1 - \frac{z}{l_2}} \\
    \Rightarrow & x' = \frac{x}{1 - \frac{z}{l_2}}
\end{align*}
\]
Perspective Projection

- First we apply a perspective transform to a point $X$ that takes it to $X'$.

\[ X' = P_r X \]

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  w'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

\[ x' = \frac{x}{w+1}, \quad y' = \frac{y}{w+1}, \quad z' = \frac{z}{w+1} \]

To find the vanishing point along the z direction we apply the perspective transformation to the point at infinity along the z direction.

\[ X' = P_r X \]

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  w'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

\[ x' = \frac{x}{w+1}, \quad y' = \frac{y}{w+1}, \quad z' = \frac{z}{w+1} \]

- If $x = -1/z$, then we get $z' = -z$, i.e., the vanishing point lies on the opposite side of the projection plane as the center of projection.

Perspective Projection

- Now we add projection on the plane $z = 0$.

\[ X' = P(z=0) P_r X \]

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  w'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

\[ x' = \frac{x}{w+1}, \quad y' = \frac{y}{w+1}, \quad z' = 0 \]

- If $x = -1/z$, then we get $x' = \frac{x}{w+1}$, $y' = \frac{y}{w+1}$, and $z' = 0$.

Perspective Projection

Single point perspective

- Centre of projection (CoP) on x axis

\[ X' = P_r X \]

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  w'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

\[ x' = \frac{x}{px+1}, \quad y' = \frac{y}{px+1}, \quad z' = \frac{z}{px+1} \]

- CoP is at $(-1/p,0,0,1)$, VP is at $(1/p,0,0,1)$

Perspective Projection

Single point perspective

- Centre of projection (CoP) on y axis

\[ X' = P_r X \]

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  w'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

\[ x' = \frac{x}{qy+1}, \quad y' = \frac{y}{qy+1}, \quad z' = \frac{z}{qy+1} \]

- CoP is at $(0,-1/q,0,1)$, VP is at $(0,1/q,0,1)$
Two point perspective

$$P_{11} = P_0 P_1$$

$$P_{11} = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}$$

$$x' = \frac{x}{px + qy + 1}, \quad y' = \frac{y}{px + qy + 1}, \quad z' = \frac{z}{px + qy + 1}$$

- Two vanishing points
- Two CoPs?

Three point perspective

$$P_{11} = P_0 P_1 P_2$$

$$P_{11} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}$$

$$x' = \frac{x}{px + qy + rz + 1}, \quad y' = \frac{y}{px + qy + rz + 1}, \quad z' = \frac{z}{px + qy + rz + 1}$$

- Three vanishing points
- Three CoPs?

Generation of perspective views

- Translation along x-y line.
- Rotate about y axis and then apply single point perspective projection.

$$T = P_1(z=0), T(l, m, n) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$
Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point perspective projection.

$$T = P_z(z=0) \cdot R_x(\phi) \cdot R_y(\theta)$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
0 & 1 & 0 & 0 \\
-\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & r & 1 \\
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \theta & -\sin \theta \\
0 & 0 & \sin \theta & \cos \theta \\
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
0 & 1 & 0 & 0 \\
-\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & r & 1 \\
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
\sin \phi \cos \theta & \cos \phi & -\sin \phi & 0 \\
0 & 0 & 1 & 0 \\
-r \cos \phi \sin \phi & r \sin \phi & r \cos \theta \cos \phi & 1 \\
\end{bmatrix}$$

- We get a two point perspective.

Perspective Projection

Generation of perspective views

- Rotate about y axis and x axis and then apply single point perspective projection.

$$T = P_z(z=0) \cdot R_y(\theta) \cdot R_x(\phi)$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
0 & 1 & 0 & 0 \\
-\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & r & 1 \\
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
\sin \phi \cos \theta & \cos \phi & -\sin \phi & 0 \\
0 & 0 & 1 & 0 \\
-r \cos \phi \sin \phi & r \sin \phi & r \cos \theta \cos \phi & 1 \\
\end{bmatrix}$$

- We get a three point perspective.

Taxonomy

Planar Projections

- Parallel
- Orthographic
- Front
- Top
- Side

Perspective

- One Point
- Two Point
- Three Point