Viewing
Perspective Projection

Centre of Projection

Image Plane
or Projection Plane

Object

Projectors
Parallel Projection

Centre of Projection?

Image Plane or Projection Plane

Object

Projectors
Parallel Projection

Orthographic Projection

- Multiviews (x=0, y=0 or z=0 or principal planes).
- True size or shape for lines.
- For projection on the z=0 plane we get the projection matrix as

\[
P(z=0) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Parallel Projection

Axonometric Projection

• Transform and then project using an orthographic projection such that at multiple adjacent faces are visible – better representation of a 3D object using 1 view. Face parallel to projection plane shows true shape and size.

• If $U$ be the matrix formed by stacking up the unit vectors along the three axes, and $T$ be the axonometric projection, then

$$T \cdot U = T \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x_x' & x_y' & x_z' \\ y_x' & y_y' & y_z' \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
Parallel Projection

Axonometric Projection

- If $U$ be the matrix formed by stacking up the unit vectors along the three axes, and $T$ be the axonometric projection, then

$$T.U = T \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ x' \\ y' \end{bmatrix}$$

- The foreshortening ratios for each projected principal axes are then given by:

$$f_x = \sqrt{x_x'^2 + y_x'^2} \quad f_y = \sqrt{x_y'^2 + y_y'^2} \quad f_z = \sqrt{x_z'^2 + y_z'^2}$$
Parallel Projection

Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
  - Trimetric (all foreshortenings are different)

\[ f_x \neq f_y \neq f_z \]
Parallel Projection

Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
  - Trimetric (all foreshortenings are different)
  - Dimetric (two foreshortenings are the same)
Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections:
  - Trimetric (all foreshortenings are different)
  - Dimetric (two foreshortenings are the same)
  - Isometric (all foreshortenings are the same)
Parallel Projection

Axonometric Projection

- Assuming we rotate by $R_y(\phi)$ and $R_x(\theta)$ before we do the projection on the $z=0$ plane.

$$T = P(z=0) \cdot R_x(\theta) \cdot R_y(\phi)$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
0 & 1 & 0 & 0 \\
-\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
\sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$
Parallel Projection

Axonometric Projection

• Now we apply this axonometric projection $T$ to $U$

$$T.U = \begin{bmatrix} 
\cos \phi & 0 & \sin \phi & 0 \\
\sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 
\cos \phi & 0 & \sin \phi \\
\sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi \\
0 & 0 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}$$
Parallel Projection

Axonometric Projection

- So the foreshortening ratios become

\[ f_x^2 = \cos^2 \phi + \sin^2 \theta \sin^2 \phi \]
\[ f_y^2 = \cos^2 \theta \]
\[ f_z^2 = \sin^2 \phi + \sin^2 \theta \cos^2 \phi \]

- For Isometric projections, if we solve for \( f_x = f_y = f_z \) then we get \( \theta = 35.26^\circ \) and \( \phi = \pm 45^\circ \)
Parallel Projection

Oblique Projection

- The projectors are parallel to each other but they are not perpendicular to the plane of projection.
- Only planes parallel to plane of projection show true shape and size.
Parallel Projection

Oblique Projection

\[ P'(x, y) \]

\[ P''(x_p, y_p) \]

\[ x_p = x + l \cos \beta \]

\[ y_p = y + l \sin \beta \]

\[ \tan \alpha = \frac{z}{l} \quad \text{or} \quad l = z \cot \alpha \]
Parallel Projection

Oblique Projection

- When $\alpha = 45^\circ$ we get a **Cavalier** projection. Lines perpendicular to the projection plane are not foreshortened.

- When $\cot \alpha = 1/2$ we get **Cabinet** projections. Lines perpendicular to the projection plane are foreshortened by half.

- $\beta$ is typically $30^\circ$ or $45^\circ$. 
Perspective Projection

- Projectors converge at a finite centre of projection.
- Parallel lines converge.
- We get non-uniform foreshortening.
- Shape is not preserved.
- We see in perspective – so perspective viewing seems natural and helps in depth perception.
A digression into art

13\textsuperscript{th} century, Arezzo by Giotto
A digression into art

Early 15th century, The Little Garden of Paradise
A digression into art

15th century, The Baptistry in Florence, Filippo Brunelleschi
15th century, Fresco of Holy Trinity, Masaccio
A digression into art

15th century, School of Athens, Raphael
Perspective Projection

\[ P(x, y, z) \quad P'(x', y') \]

\[ \frac{y'}{l_2} = \frac{y}{l_2 - l_1} \quad \frac{z_c}{l_2} = \frac{z_c - z}{l_2 - l_1} \]

\[ \Rightarrow y' = \frac{y}{1 - \frac{z}{z_c}} \]
Perspective Projection

Perspective projection is a type of projection where parallel lines converge at a single point. In this projection, the object is perceived as if viewed through a camera lens or a pinhole, and the rays extend infinitely from the viewpoint to the image plane.

Mathematically, the projection of a point $P(x, y, z)$ to its perspective projection $P'(x', y', z')$ can be represented as:

$$\frac{x'}{l_2} = \frac{x}{l_2 - l_1} \quad \text{and} \quad \frac{z_c}{l_2} = \frac{z_c - z}{l_2 - l_1}$$

Simplifying the first equation gives:

$$x' = \frac{x}{1 - \frac{z}{z_c}}$$

The diagram illustrates this concept with a perspective view, showing how the projections of points in 3D space are transformed into 2D space, with the camera lens or pinhole located at the center of projection.
Perspective Projection

- First we apply a perspective transform to a point \( x \) that takes it to \( x' \)

\[
X' = P_r \cdot X
\]

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
w'
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & r & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix} =
\begin{bmatrix}
x \\
y \\
z \\
rz+1
\end{bmatrix}
\]

\[
x' = \frac{x}{rz+1}, \quad y' = \frac{y}{rz+1}, \quad z' = \frac{z}{rz+1}
\]
Perspective Projection

- First we apply a perspective transform to a point $X$ that takes it to $X'$.
- Now we add projection on the $z=0$ plane.

$$X' = P(z=0) \cdot P_r \cdot X$$

$$\begin{bmatrix}
    x' \\
    y' \\
    z' \\
    w'
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \cdot
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & r & 1
\end{bmatrix} \cdot
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix} =
\begin{bmatrix}
    x \\
    y \\
    0 \\
    rz+1
\end{bmatrix}$$

$$x' = \frac{x}{rz+1}, \quad y' = \frac{y}{rz+1}, \quad z' = 0$$

- If $r = -\frac{1}{z_c}$ then we get

$$x' = \frac{x}{1 - \frac{z}{z_c}}, \quad y' = \frac{y}{1 - \frac{z}{z_c}}$$
Perspective Projection

- Vanishing point in the z direction.
- Set of lines not parallel to the projection plane converge at a vanishing point.
Perspective Projection

- To find the vanishing point along the z direction we apply the perspective transformation to the point at infinity along the z direction.

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  w'
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & r & 1
\end{bmatrix}
\begin{bmatrix}
  0 \\
  0 \\
  1 \\
  0
\end{bmatrix}
\]

\[
x' = 0, \quad y' = 0, \quad z' = \frac{1}{r}
\]

- If \( r = -\frac{1}{z_c} \) then we get \( z' = -z_c \), i.e., the vanishing point lies an equal distance on the opposite side of the projection plane as the center of projection.
Perspective Projection

Single point perspective

- Centre of projection (CoP) on x axis

\[
\begin{bmatrix}
    x' \\
    y' \\
    z' \\
    w'
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    p & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix} = \begin{bmatrix}
    x \\
    y \\
    z \\
    px + 1
\end{bmatrix}
\]

\[
x' = \frac{x}{px + 1}, \quad y' = \frac{y}{px + 1}, \quad z' = \frac{z}{px + 1}
\]

- CoP is at \((-1/p, 0, 0, 1)\), VP is at \((1/p, 0, 0, 1)\)
Perspective Projection

Single point perspective

- Centre of projection (CoP) on y axis

\[
X' = P_q \cdot X
\]

\[
\begin{bmatrix}
    x' \\
    y' \\
    z' \\
    w'
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & q & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix} =
\begin{bmatrix}
    x \\
    y \\
    z \\
    qy + 1
\end{bmatrix}
\]

\[
x' = \frac{x}{qy + 1}, \quad y' = \frac{y}{qy + 1}, \quad z' = \frac{z}{qy + 1}
\]

- CoP is at \((0, -1/q, 0, 1)\), VP is at \((0, 1/q, 0, 1)\)
Perspective Projection

Two point perspective

\[ P_{pq} = P_p \cdot P_q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & 0 & 1 \end{bmatrix} \]

\[ x' = \frac{x}{px + qy + 1}, \quad y' = \frac{y}{px + qy + 1}, \quad z' = \frac{z}{px + qy + 1} \]

- Two vanishing points
- Two CoPs?
Perspective Projection

Three point perspective

\[ P_{pqr} = P_p \cdot P_q \cdot P_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & r & 1 \end{bmatrix} \]

\[ x' = \frac{x}{px + qy + rz + 1}, \quad y' = \frac{y}{px + qy + rz + 1}, \quad z' = \frac{z}{px + qy + rz + 1} \]

- Three vanishing points
- Three CoPs?

M. C. Esher, Ascending and Descending
Perspective Projection

Generation of perspective views

- Transform and then apply single point perspective.
- Let us try to translate, apply a perspective and project to z=0

\[
T = P_r(z=0). T(l, m, n) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & r & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & 0 & l \\
0 & 1 & 0 & m \\
0 & 0 & 1 & n \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & l \\
0 & 1 & 0 & m \\
0 & 0 & 0 & 0 \\
0 & 0 & r & rn+1
\end{bmatrix}
\]
Perspective Projection

Generation of perspective views

- Translation along x=y line.
Translation along the z axis causes change in scaling.
Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point perspective projection.

\[ T = P_z(z=0). \quad R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point perspective projection.

\[ T = P_{r}(z=0) \cdot R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r \sin \phi & 0 & r \cos \phi & 1 \end{bmatrix} \]

- We get a two point perspective.
Perspective Projection

Generation of perspective views

- Rotate about y axis, x axis and then apply single point perspective projection.

\[ T = P_z(z=0) \cdot R_x(\theta) \cdot R_y(\phi) \]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & r \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos\theta & -\sin\theta & 0 \\
0 & \sin\theta & \cos\theta & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\cos\phi & 0 & \sin\phi & 0 \\
0 & 1 & 0 & 0 \\
-\sin\phi & 0 & \cos\phi & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Perspective Projection

Generation of perspective views

- Rotate about y axis, x axis and then apply single point perspective projection.

\[
T = P_r(z=0) \cdot R_x(\theta) \cdot R_y(\phi)
\]

\[
= \begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
\sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\
0 & 0 & 0 & 0 \\
-r \cos \theta \sin \phi & r \sin \theta & r \cos \theta \cos \phi & 1
\end{bmatrix}
\]

- We get a three point perspective.
Planar Projections

Parallel
- Orthographic
  - Front
  - Top
  - Side
- Axonometric
  - Trimetric
  - Dimetric
  - Isometric
- Oblique
  - Cavalier
  - Cabinet

Perspective
- One Point
- Two Point
- Three Point