Viewing
Perspective Projection

Centre of Projection

Image Plane or Projection Plane

Object

Projectors
Parallel Projection

Centre of Projection?

Object

Projectors

Image Plane or Projection Plane
Parallel Projection

Orthographic Projection

- Multiviews (x=0, y=0 or z=0 or principal planes).
- True size or shape for lines.
- For projection on the z=0 plane we get the projection matrix as

\[
P(z=0) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Parallel Projection

Axonometric Projection

- Transform and then project using an orthographic projection such that at multiple adjacent faces are visible – better representation of a 3D object using 1 view. Face parallel to projection plane shows true shape and size.

- If $U$ be the matrix formed by stacking up the unit vectors along the three axes, and $T$ be the axonometric projection, then

$$T.U = T.$$

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_x' \\
x_y' \\
x_z'
\end{bmatrix}
=
\begin{bmatrix}
x_x' & x_y' & x_z' \\
y_x' & y_y' & y_z' \\
0 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
\]
Parallel Projection

Axonometric Projection

- If $U$ be the matrix formed by stacking up the unit vectors along the three axes, and $T$ be the axonometric projection, then

$$T . U = T . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x'_x & x'_y & x'_z \\ y'_x & y'_y & y'_z \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

- The foreshortening ratios for each projected principal axes are then given by:

$$f_x = \sqrt{x'_x^2 + y'_x^2} \quad f_y = \sqrt{x'_y^2 + y'_y^2} \quad f_z = \sqrt{x'_z^2 + y'_z^2}$$
Parallel Projection

Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
  - Trimetric (all foreshortenings are different)
Parallel Projection

Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections
  - Trimetric (all foreshortenings are different)
  - Dimetric (two foreshortenings are the same)

\[ f_x \neq f_y \neq f_z \]

\[ f_x = f_z \]
Parallel Projection

Axonometric Projection

- Depending on the kind of foreshortening they cause we can have three types of axonometric projections:
  - Trimetric (all foreshortenings are different)
  - Dimetric (two foreshortenings are the same)
  - Isometric (all foreshortenings are the same)
Parallel Projection

Axonometric Projection

- Assuming we rotate by $R_y(\phi)$ and $R_x(\theta)$ before we do the projection on the $z=0$ plane.

$$T = P(z=0) \cdot R_x(\theta) \cdot R_y(\phi)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ \sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Parallel Projection

Axonometric Projection

- Now we apply this axonometric projection $T$ to $U$

\[
T.U = \begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
\sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
1 \\
\end{bmatrix} = \begin{bmatrix}
\cos \phi & 0 & \sin \phi \\
\sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi \\
0 & 0 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}
\]
Parallel Projection

Axonometric Projection

- So the foreshortening ratios become

\[ f_x^2 = \cos^2 \phi + \sin^2 \theta \sin^2 \phi \]
\[ f_y^2 = \cos^2 \theta \]
\[ f_z^2 = \sin^2 \phi + \sin^2 \theta \cos^2 \phi \]

- For Isometric projections, if we solve for \( f_x = f_y = f_z \) then we get

\[ \theta = 35.26^\circ \] and \[ \phi = \pm 45^\circ \]
Parallel Projection

Oblique Projection

- The projectors are parallel to each other but they are not perpendicular to the plane of projection.
- Only planes parallel to plane of projection show true shape and size.
Parallel Projection

Oblique Projection

\[ P'(x, y) \]

\[ P''(x_p, y_p) \]

\[ P'''(x_p, y_p) \]

\[ \beta \]

\[ \alpha \]

\[ x_p = x + l \cos \beta \]

\[ y_p = y + l \sin \beta \]

\[ \tan \alpha = \frac{z}{l} \]

or

\[ l = z \cot \alpha \]
When we get a **Cavalier** projection. Lines perpendicular to the projection plane are not foreshortened.

- **Oblique Projection**

- When $\alpha = 45^\circ$ we get a **Cavalier** projection. Lines perpendicular to the projection plane are not foreshortened.

- When $\cot \alpha = 1/2$ we get **Cabinet** projections. Lines perpendicular to the projection plane are foreshortened by half.

- $\beta$ is typically $30^\circ$ or $45^\circ$. 

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CS475/CS675: Lecture 6
Perspective Projection

- Projectors converge at a finite centre of projection.
- Parallel lines converge.
- We get non-uniform foreshortening.
- Shape is not preserved.
- We see in perspective – so perspective viewing seems natural and helps in depth perception.
A digression into art

13th century, Arezzo by Giotto
A digression into art

Early 15th century, The Little Garden of Paradise
A digression into art

15th century, The Baptistry in Florence, Filippo Brunelleschi
15th century, Fresco of Holy Trinity, Masaccio
15th century, School of Athens, Raphael
Perspective Projection

\[ P(x, y, z) \]

\[ P'(x', y') \]

\[
\frac{y'}{l_2} = \frac{y}{l_2 - l_1} \quad \frac{z_c}{l_2} = \frac{z_c - z}{l_2 - l_1}
\]

\[ \Rightarrow y' = \frac{y}{1 - \frac{z}{z_c}} \]
Perspective Projection

\[ P(x, y, z) \]

\[ P'(x', y') \]

\[ \frac{x'}{l_2} = \frac{x}{l_2 - l_1} \]

\[ \frac{z_c}{l_2} = \frac{z_c - z}{l_2 - l_1} \]

\[ \Rightarrow x' = \frac{x}{1 - \frac{z}{z_c}} \]
Perspective Projection

- First we apply a perspective transform to a point \( x \) that takes it to \( x' \)

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
w'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & r & 1
\end{bmatrix} \cdot \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix} = \begin{bmatrix}
x \\
y \\
z \\
rz + 1
\end{bmatrix}
\]

\[
x' = \frac{x}{rz + 1}, \quad y' = \frac{y}{rz + 1}, \quad z' = \frac{z}{rz + 1}
\]
Perspective Projection

- First we apply a perspective transform to a point $X$ that takes it to $X'$.
- Now we add projection on the $z = 0$ plane.

$$X' = P(z=0) \cdot P_r \cdot X$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ rz + 1 \end{bmatrix}$$

$$x' = \frac{x}{rz + 1}, \quad y' = \frac{y}{rz + 1}, \quad z' = 0$$

- If $r = \frac{-1}{z_c}$ then we get

$$x' = \frac{x}{1 - \frac{z}{z_c}}, \quad y' = \frac{y}{1 - \frac{z}{z_c}}$$
Perspective Projection

- Vanishing point in the z direction.
- Set of lines not parallel to the projection plane converge at a vanishing point.

$VP_z$
Perspective Projection

- To find the vanishing point along the z direction we apply the perspective transformation to the point at infinity along the z direction.

\[
X' = P_r \cdot X
\]

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
w'
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & r & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
1 \\
r
\end{bmatrix}
\]

\[x' = 0, \quad y' = 0, \quad z' = \frac{1}{r}\]

- If \( r = -1/z_c \) then we get \( z' = -z_c \), i.e., the vanishing point lies an equal distance on the opposite side of the projection plane as the center of projection.
Perspective Projection

Single point perspective

- Centre of projection (CoP) on x axis

\[
X' = P_p \cdot X
\]

\[
\begin{bmatrix}
x' \\
y' \\
z' \\
w'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
p & 0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix} = \begin{bmatrix}
x \\
y \\
z \\
px + 1
\end{bmatrix}
\]

\[
x' = \frac{x}{p x + 1}, \quad y' = \frac{y}{p x + 1}, \quad z' = \frac{z}{p x + 1}
\]

- CoP is at \((- \frac{1}{p}, 0, 0, 1\)) ,VP is at \((\frac{1}{p}, 0, 0, 1)\)
Perspective Projection

Single point perspective

- Centre of projection (CoP) on y axis

\[ X' = P_q \cdot X \]

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  w'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & q & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix} =
\begin{bmatrix}
  x \\
  y \\
  z \\
  qy + 1
\end{bmatrix}
\]

\[
x' = \frac{x}{qy + 1}, \quad y' = \frac{y}{qy + 1}, \quad z' = \frac{z}{qy + 1}
\]

- CoP is at \((0, -1/q, 0, 1)\), VP is at \((0, 1/q, 0, 1)\)
Perspective Projection

Two point perspective

$$P_{pq} = P_p \cdot P_q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & 0 & 1 \end{bmatrix}$$

$$x' = \frac{x}{px + qy + 1}, \quad y' = \frac{y}{px + qy + 1}, \quad z' = \frac{z}{px + qy + 1}$$

- Two vanishing points
- Two CoPs?
Perspective Projection

Three point perspective

\[ P_{pqr} = P_p \cdot P_q \cdot P_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & q & r & 1 \end{bmatrix} \]

\[ x' = \frac{x}{px + qy + rz + 1}, \quad y' = \frac{y}{px + qy + rz + 1} \]

\[ z' = \frac{z}{px + qy + rz + 1} \]

- Three vanishing points
- Three CoPs?

CS475/CS675: Lecture 6
Perspective Projection

Generation of perspective views

- Transform and then apply single point perspective.
- Let us try to translate, apply a perspective and project to $z=0$

$$T = P_r(z=0). \quad T(l, m, n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & rn+1 \end{bmatrix}$$
Perspective Projection

Generation of perspective views

- Translation along x=y line.
Perspective Projection

Generation of perspective views

- Translation along the z axis causes change in scaling.
Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point perspective projection.

\[ T = P_y(z=0), \quad R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix}, \quad \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
Perspective Projection

Generation of perspective views

- Rotate about y axis and then apply single point perspective projection.

\[ T = P_z(z = 0) \cdot R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r \sin \phi & 0 & r \cos \phi & 1 \end{bmatrix} \]

- We get a two point perspective.
Perspective Projection

Generation of perspective views

- Rotate about y axis, x axis and then apply single point perspective projection.

\[ T = P_r(z = 0) \cdot R_x(\theta) \cdot R_y(\phi) \]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & r & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
0 & 1 & 0 & 0 \\
-\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Perspective Projection

Generation of perspective views

- Rotate about y axis, x axis and then apply single point perspective projection.

\[ T = P_r(z=0) \cdot R_x(\theta) \cdot R_y(\phi) \]

\[
\begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
\sin \theta \sin \phi & \cos \theta & -\sin \theta \cos \phi & 0 \\
0 & 0 & 0 & 0 \\
-r \cos \theta \sin \phi & r \sin \theta & r \cos \theta \cos \phi & 1
\end{bmatrix}
\]

- We get a three point perspective.