Perspective Projection

Parallel Projection

Parallel Projection

Orthographic Projection

Axonometric Projection

Axonometric Projection

The foreshortening ratios for each projected principal axes are then given by:

- For parallel projection:
  
  \[ f_x = \frac{1}{2} \left( x^2 + y^2 \right) \]

- For orthographic projection:
  
  \[ f_x = \frac{1}{2} \left( x^2 + y^2 \right) \]

For parallel projection:

- Transform and then project using an orthographic projection such that at multiple adjacent faces are visible - better representation of a 3D object using 1 view. Face parallel to projection plane shows true shape and size.

- If \( T \) be the matrix formed by stacking up the unit vectors along the three axes, and \( T' \) be the axonometric projection, then

\[ T' = T \cdot \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
x' & y' & z' \\
x'' & y'' & z'' \\
0 & 0 & 1
\end{bmatrix} \]
Parallel Projection

Axonometric Projection

Depending on the kind of foreshortening they cause we can have three types of axonometric projections:

- Trimetric (all foreshortenings are different)
  \[ f_x \neq f_y \neq f_z \]

- Dimetric (two foreshortenings are the same)
  \[ f_x = f_z \]

- Isometric (all foreshortenings are the same)
  \[ f_x = f_y = f_z \]

Assuming we rotate by \( \theta \) and \( \phi \) before we do the projection on the \( z=0 \) plane.

\[
T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cos \phi & \sin \phi & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin \phi \cos \theta & \cos \phi \cos \theta & \sin \phi \cos \theta & 0 \end{pmatrix}
\]

Now we apply this axonometric projection \( T \) to \( U \).

\[
T U = \begin{pmatrix} \cos \phi & 0 & \sin \phi & 0 \\ \sin \phi & 0 & -\cos \phi & 0 \\ \cos \phi & 0 & \sin \phi & 0 \\ \sin \phi & 0 & -\cos \phi & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

So the foreshortening ratios become

\[ f_x' = \cos^2 \phi + \sin^2 \theta \sin^2 \phi \]
\[ f_y' = \cos^2 \theta \]
\[ f_z' = \sin^2 \phi + \sin^2 \theta \cos^2 \phi \]

For Isometric projections, if we solve for \( f_x = f_y = f_z \), then we get

\( \theta = 35.26^\circ \) and \( \phi = 45^\circ \).
Parallel Projection

- The projectors are parallel to each other but they are not perpendicular to the plane of projection.
- Only planes parallel to plane of projection show true shape and size.

\[
\begin{align*}
\mathbf{P}(x, y, z) & \rightarrow \mathbf{P'}(x', y', z') \\
& = (x + l \cos \beta, y + l \sin \beta, z)
\end{align*}
\]

\[l \cot \alpha = \frac{1}{2} \quad \text{or} \quad l = z \cot \alpha\]

Oblique Projection

- When \[\alpha = 45^\circ\] we get a **Cavalier** projection. Lines perpendicular to the projection plane are not foreshortened.
- When \[\cos \alpha = \frac{1}{2}\] we get **Cabinet** projections. Lines perpendicular to the projection plane are foreshortened by half.

- \(\alpha\) is typically \(30^\circ\) or \(45^\circ\).

Perspective Projection

- Projectors converge at a finite centre of projection.
- Parallel lines converge.
- We get non-uniform foreshortening.
- Shape is not preserved.
- We see in perspective – so perspective viewing seems natural and helps in depth perception.

A digression into art

13th century, Arezzo by Giotto

Early 15th century, The Little Garden of Paradise
15th century, The Baptistry in Florence, Filippo Brunelleschi.

Fresco of Holy Trinity, Masaccio.

School of Athens, Raphael.

Perspective Projection

First we apply a perspective transform to a point $X$ that takes it to $X'$.

\[
\begin{bmatrix}
 x' \\
 y' \\
 z' \\
 w'
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 x \\
 y \\
 z \\
 w
\end{bmatrix}
\]

\[
X' = P X
\]

\[
\begin{bmatrix}
 x' \\
 y' \\
 z' \\
 w'
\end{bmatrix} =
\begin{bmatrix}
 x \\
 y \\
 z \\
 1
\end{bmatrix}
\]

\[
X' = \frac{\begin{bmatrix}
 x' \\
 y' \\
 z'
\end{bmatrix}}{w'}
\]

\[
X' = \begin{bmatrix}
 x' \\
 y' \\
 z'
\end{bmatrix}
\]

\[
\begin{bmatrix}
 x \\
 y \\
 z \\
 w
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 x' \\
 y' \\
 z'
\end{bmatrix}
\]

\[
X' = \frac{\begin{bmatrix}
 x' \\
 y' \\
 z'
\end{bmatrix}}{w'}
\]
Perspective Projection

Single point perspective

- Centre of projection (CoP) on x axis

\[
\begin{bmatrix}
X' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

- CoP is at (1, 0, 0, 1), VP is at (0, 0, 0, 1)

CS475/CS675: Lecture 6

Two point perspective

- Centre of projection (CoP) on y axis

\[
\begin{bmatrix}
X' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

- CoP is at (0, 1, 0, 1), VP is at (1, 0, 0, 1)

CS475/CS675: Lecture 6

Perspective Projection

Single point perspective

- Centre of projection (CoP) on y axis

\[
\begin{bmatrix}
X' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

- CoP is at (0, 1, 0, 1), VP is at (1, 0, 0, 1)

CS475/CS675: Lecture 6

Perspective Projection

Two point perspective

- Centre of projection (CoP) on z axis

\[
\begin{bmatrix}
X' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

- CoP is at (0, 0, 1, 0), VP is at (0, 0, 1, 1)

CS475/CS675: Lecture 6

Perspective Projection

- Centre of projection (CoP) on y axis

\[
\begin{bmatrix}
X' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

- CoP is at (0, 1, 0, 1), VP is at (1, 0, 0, 1)

CS475/CS675: Lecture 6
Perspective Projection

Three point perspective

\[ P_{pq} = P_p \cdot P_q \cdot P_r \]

\[ x' = \frac{x}{x_h + a \cdot z + b} \]
\[ y' = \frac{y}{x_h + a \cdot z + b} \]

Three vanishing points

Three CoPs?

Perspective Projection

Generation of perspective views

- Translation and then apply single point perspective.
- Let us try to translate, apply a perspective and project to \( z = 0 \).

\[ T = P(z = 0), T(l, m, n) = [1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1] \]

- Translation along the \( x = y \) line.
- Translation along the \( z \) axis causes change in scaling.

Perspective Projection

Generation of perspective views

- Rotate about \( y \) axis and then apply single point perspective projection.

\[ T = P(z = 0), R_y(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \phi & 0 & -\sin \phi & 0 & 0 & 0 & \sin \phi & 0 & \cos \phi \end{bmatrix} \]

- We get a two point perspective.

Perspective Projection

Generation of perspective views

- Rotate about \( x \) axis and then apply single point perspective projection.

\[ T = P(z = 0), R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \phi & 0 & 0 & 0 & 0 \end{bmatrix} \]

- Translation along \( a \) \( b \) line.
Perspective Projection

Generation of perspective views

- Rotate about y axis, x axis and then apply single point perspective projection.

\[ T = P_{z(\varepsilon=0)} \cdot R_y(\beta) \cdot R_x(\alpha) \]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos\phi & 0 & \sin\phi \cos\beta & 0 \\
0 & 1 & 0 & 0 \\
-\sin\phi \cos\beta & 0 & \cos\phi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

We get a three point perspective.