**Problem 1:**[15 marks] Let $\pi$ be any permutation from $\{0,\ldots,2^n-1\}$ to $\{0,\ldots,2^n-1\}$. Show that it is possible to establish paths in $Q_n$ from $i$ to $\pi(i)$ for all $i$ such that at most 2 paths pass through any edge in each direction. The length of the paths may be larger than $n$.

This can be done with congestion 1 on a $2^n$ input Benes network. A Benes network is simply a butterfly network connected to its reverse. A single butterfly when collapsed into a column gives a hypercube in instead of a single edge we have an edge in each direction, arising from cross edges of the Butterfly (considered directed left to right). The Benes network will therefore also give a hypercube when collapsed, with two edges in each direction. Since $\pi$ embeds with congestion 1 in the Benes, it embeds with congestion 1 in the hypercube with 2 edges in each direction. So in a standard hypercube this will cause congestion 2 in each direction.

Note: Nothing more need be written.

**Problem 2:** We showed in class that $C_{2^n}$ can be embedded into $Q_n$ with load = dilation = 1. The base case is: $C_4 = Q_2$, with cycle nodes 0,1,2,3 placed at hypercube nodes 0,1,3,2. We take two cycles $r, s$ embedded (inductively) in copies $R, S$ of $Q_{n-1}$. We then remove the edges $(2^{n-1}-1, 0)$ of $r, s$ and connect nodes $0, 2^{n-1}-1$ of $r$ to corresponding nodes of $s$ and get the bigger cycle $t$ in the bigger hypercube $T$. The numbering of $t$ is obtained by extending the numbering of $r$, i.e. nodes 0,1 of $r$ are numbered 0,1 for $t$ as well and the subsequent nodes are numbered accordingly.

(a) [5 marks] Consider the Lemma: For $1 \le z < 2^n - 1$, edge $(z-1, z)$ of $t$ crosses dimension $j$ of $T$ if and only if $z \bmod 2^{j+1} = 2^j$, i.e. the least significant $j$ bits of $z$ are 0, and the next least significant bit is 1. Prove the Lemma for $j = 0$. First state it as simply as possible for $j = 0$.

All edges of the type $(2k, 2k+1)$ cross dimension 0. True for $Q_2$. Assume true for $r$, $s$. The edges removed, $2^{n-1}-1$ to 0 in both, do not use dimension 0. When we merge $r, s$, we retain all edges crossing dimension 0, and they are alternate, and (0,1) is along 0. So the rest must also have the form $(2k, 2k+1)$.

(b)[10 marks] Assuming the above Lemma (for all $j$) show that cycle nodes $k, k+2^j$ are at a distance at most 2 in the hypercube for any $j, k$. (Hint: what dimensions are traversed an odd number of times while travelling between these nodes along the cycle?)

The edges to be considered are $(z-1, z)$ where $z = k2^j + 1$ through $(k+1)2^j$. The second endpoint takes $2^j$ consecutive values along the path.

Of these, half are odd. Thus by part (a) the corresponding edges would have used dimension 0.

Half of the rest end with 10, and they would use dimension 1.

In general the pattern $2^k$ appears $2^{j-k-1}$ times, for $k < j$. Thus we will have an even number of traversals of dimension $k$ for $j-k-1 \ge 1$, i.e. $k \le j-2$. Thus there can be an odd number of traversals for $k = j-1$. For $k = j$ we will have the path cross a dimension $j$ or higher, and hence even that will be odd.

But $k, k+2^j$ will not differ in dimensions which are crossed an even number of times. So at most they will be different in 2 dimensions, i.e. will be at distance 2.

(c)[10 marks] The X tree, $X_n$ is obtained from the complete binary tree $T_n$ on $2^n - 1$ nodes as follows. Imagine that $T_n$ is drawn in the standard manner, with the root at the top in level 0, and leaves in $n-1$. Level $i$ has $2^i$ vertices, and we connect them using a path $P_{2^i}$ going left to right for each $i$. Show that the X-tree has a load 1, dilation 2 embedding in $Q_n$. Hint: use part (d).

Number the nodes 0 through $2^n - 1$ left to right. Embed node numbered $i$ at the place where cycle node $i$ is kept as per the Gray code embedding above. Now nodes across tree edges are $(x, x+1)$ and hence they are neighbours in $Q_n$. Horizontal neighbours differ by some power $2^j$ and hence are a distance 2 apart.

A nice inductive proof was also given by several. The induction hypothesis is: the $2^n - 1$ node X-tree can be embedded in $Q_n$ with dilation 2 such that the unused node is adjacent to the root. Now when you take 2 copies of $Q_{n-1}$ you make one of the unused nodes be the root, which will be adjacent to the other unused node, as required. Note that the new root will connect with a dilation of 2 to the old root. The base case is easy, say $Q_2$.

(d)[10 marks] Give a tree decomposition for $X_4$ of width 3.

Eliminate the root and extreme leaf nodes. Find the tree embedding for the remaining graph by trial and error, and then add in 3 edges for the 3 eliminated nodes. Many solutions possible.

(e)[5 marks] Is $X_4$ a chordal graph? Justify your answer. It is not. (2 marks). This is because there is no perfect elimination order (1 mark). Why: After eliminating the root and the extreme left and right leaves, nothing can be eliminated without introducing fill in. (2 marks).

**Problem 3:** In the Hamiltonian Cycle problem we are given an $n$ vertex undirected graph $G$ and the goal is to determine if the cycle on $n$ vertices is a subgraph of $G$, and if so, find it. Suppose now that $G$ has a known tree decomposition $(T, f)$ of width $k$ (constant).

In the questions below you will work out the key ingredients of the algorithm to find a Hamiltonian cycle for $G$ given $(T, f)$. $T$ is rooted at some $r$. Remember that for a node $u$ of $T$, $V_u =$ set of vertices of $G$ whose subtrees contain $u$. $T(u) =$ subtree of $T$ under $u$. $V_{T(u)} =$ set of vertices of $G$ whose subtree contains any node in $T(u)$. $H_u =$ graph induced in $G$ by $V_{T(u)}$, and $G_u =$ the graph induced by $V_{T(u)} - V_u$.

(a)[5 marks] Suppose $X$ is a length $n$ cycle that is a subgraph of $G$. Precisely characterize the portions $X_u$ of $X$ that intersect with $H_u$.

$X_u =$ Partition of the vertices of $H_u$ into paths that start and end on $V_u$.

(b)[5 marks] Based on part (a), what boundary constrained problem will you solve on $H_u$ so as to help find a Hamiltonian cycle in $G$? Dont worry if this problem doesnt resemble Hamiltonian cycle. Clearly state the input to the problem including the role played by the boundary vertices (those in $V_u$).

Input: a list of pairs of vertices $s_i, t_i \in V_u$ such that a path must be constructed in $H_u$ from $s_i$ to $t_i$. Each vertex in $V_u$ should appear at least once, and at most once as $s_i$ and at most once as a $t_j$. All vertices of $G_u$ must be used exactly once in the paths. This will have to be solved for all valid inputs.

(c)[5 marks] Give an upper bound on the number of all possible instances of the problem in (b). This is expected to be a function only of $|V_u|$, however large.

At most each $v \in V_u$ is a source for some path and there are $|V_u|$ choices for each. So an upper bound is $|V_u|^{|V_u|}$. This is reasonably good, because at least half the nodes will be sources. The $(s, t)$ pairs also can be thought of as directed edges in a $|V_u|$ vertex graph. So the input is simply a graph, and the number of all possible graphs is $2^{|V_u|^2}$. Credit given to anything between these, with reasonable explanation.

(d)[10 marks] Suppose $u$ has exactly two children $u_1, u_2$ and suppose that $V_u = V_{u_1} = V_{u_2}$ (yes, this is redundant, but never mind). Show how the solutions to instances $u_1, u_2$ on $H_{u_1}, H_{u_2}$ can be combined to get a solution to some instance, if any, on $H_u$. Describe the instance.

A brief answer giving the gist of the idea was expected. Something like:

For each subtree we will solve a number of instances of the boundary constrained problem. Suppose $g_1, g_2$ are instances for subtrees $T(u_1), T(u_2)$, and each instance has a yes answer. Then if $g_1, g_2$ are edge disjoint graphs (as per the solution to (c)), then their union has a solution in the subtree of $T(u)$. If for example both $g_1, g_2$ contain the same edge, then in both subtrees we are asking for a path to be constructed, this is not possible, and hence the two instances cannot be combined. There are other cases also, but a very short description with one case would have got credit.

**Problem 4:**[20 marks] A $(\alpha, \beta, N, d)$ concentrator has two sets of vertices, $U, V$, with $|U| = N$ and $|V| = N/2$. For this exercise suppose that the degree of every node in $U$ is exactly $d$, and that in $V$ is exactly $2d$. Further every subset of $U$ having $k \leq \alpha N$ nodes is connected to at least $\beta k$ nodes in $V$. Suppose we are also given that $\alpha, \beta$ are constants such that $\beta > 1, \alpha\beta > 1/4$.

Show that the diameter of this graph is $O(\log N)$.

Consider any pair of nodes $u, w$ in set $U$. We will show that there is a path of short length connecting them. For this we will start at $u$ and look at its neighbours $U_1$ in $V$, their neighbours $U_2$ in $U$ and so on. Similarly $W_1, W_2, \ldots$ for $w$. We will show that the sets $U_{2k+1}, W_{2k+1}$ must have a common vertex for some small $k$.

Consider any $U_{2k}$. The base case is $U_0 = \{u\}$. If $|U_{2k}| \leq \alpha N$, then $|U_{2k+1}| \geq \beta |U_{2k}|$. Next, the degree of every vertex in $U_{2k+1}$ is $2d$, while that of every vertex in $U_{2k+2}$ is $d$. Thus $|U_{2k+2}| \geq 2|U_{2k+1}|$. Thus if $|U_{2k}| \leq \alpha N$, then $|U_{2k+2}| \geq 2\beta |U_{2k}|$, i.e. the size of the neighbourhood 2 away has increased by a factor $2\beta$. Thus with $j = O(\log_{2\beta} \alpha N)$ we will have $|U_{2j}| \geq \alpha N$. We will consider a subset of this of size exactly $\alpha N$, and even these will have $\alpha\beta N$ neighbours in the set $U_{2j+1}$. 10 marks

Likewise we will get $|W_{2j+1}| \geq \alpha\beta N$. If these sets are to be disjoint we will need $2\alpha\beta \geq N/2 = |V|$. But we have $\alpha\beta > 1/4$, and hence this is not possible. Thus we have a path of length $O(\log N)$ between any two $u, w$. 10 marks.