A graph is planar if it can be drawn in the plane without edges crossing. More formally, a graph is planar if it has an embedding in the plane, in which each vertex is mapped to a distinct point $P(v)$, and edge (u, v) to simple curves connecting $P(u), P(v)$, such that curves intersect only at their endpoints. Examples of planar graphs: P_n, Trees, Cycles, X-tree, K_4. Examples of non-planar graphs: Q_n for n¿3, K_5, $K_{3,3}$, the complete bipartite graph which each partition having 3 vertices.

An important notion for planar graphs is that of a *Face:* which is simply a region of the plane bounded by edges of the graph. The outer infinite region is also considered a face.

Planar graphs are important for several reasons. First, they are very closely linked to the early history of graph theory. Second, in the mechanical analysis of two dimensional structures, the structures get partitioned and these partitions can be represented using planar graphs. Planar graphs are also interesting because they are a large class of graphs having small separators.

After studying some basic notions, we will study the colouring and separation of planar graphs.

1 Euler's Formula

One of the earliest results in Graph Theory is Euler's formula.

Theorem 1 (Euler's Formula) *If a finite, connected, planar graph is drawn in the plane without any edge intersections, and v is the number of vertices, e is the number of edges and f is the number of faces, then $v + f = e + 2$*

Proof: Let us generalize it to allow multiple connected components c. In that case the formula becomes $v + f = e + c + 1$. The proof is by induction over e. If $e = 0$, we have $v = c$, $f = 1$, and the theorem holds.

Suppose we remove an edge: (1) either the number of faces reduces by 1, or (2) number of components increases by 1. In each case, if the formula is true for the new graph, it is true for the old one. ■

Planar graphs are sparse.

Corollary 1 *In a finite, connected, simple, planar graph, $e \leq 3v - 6$ if $v \geq 3$. If the graph is also bipartite, then $e \leq 2v - 4$.*

Proof: If the graph is simple, then every face has at least 3 edges. Now $3f$ would count every edge 2 times, so we have $3f \leq 2e$. But $e + 2 = v + f \leq v + 2e/3$. So $3e + 6 \leq 3v + 2e$. So $e \leq 3v - 6$. In a bipartite graph every face must have at least 4 sides. Thus $4f \leq 2e$, and the result follows similarly. ■

2 Characterizing Planar Graphs

The preceding corollary already allows us to prove that K_5 and $K_{3,3}$ are not planar. For K_5 we have $v = 5$, $e = 10$, and $e \leq 3v - 6$ is not satisfied. For $K_{3,3}$ we have $v = 6, e = 9$, and $e \leq 2v - 4$ is not satisfied. It turns out that in some sense that K_5 and $K_{3,3}$ represent fundamental obstacles to planarity.

A *subdivision* of a graph G is a graph G' obtained by inserting vertices into edges of G zero or more times, e.g. A path is a subdivision of an edge.

Theorem 2 (Kuratowski's Theorem) *: A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Note that we already proved the "only if" part. The "if" part can be proved in many ways, including a constructive proof in which we find an embedding into the plane. We will not be seeing this.

G is a *minor* of H if G can be obtained from H by contracting one or more edges.

Theorem 3 (Wagner's Theorem) *A finite graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

This is very related to Kuratowski's theorem. Note that if H contains a subdivision of G, then G is a minor of H, but not vice versa.
Exercise: Show that $C_3 \square C_3$ is non planar.

Instead of drawing on the plane, we may also draw the graphs on a sphere. Indeed, by using a stereographic projection (place the sphere with its south pole on the plane, and map every point p on the plane to the intersection of the sphere surface and the line joining p to the north pole), we can see that a graph can be drawn on the sphere without crossings iff it can be drawn on the plane.

It might be interesting to note that Euler's formula applies to convex polyhedra, in which vertices, edges, faces are defined in the colloquial manner. To see this, take a point inside the polyhedron and project the vertices and edges from this point onto a sphere enclosing the polyhedron. The projections will form a planar graph.

3 Duals, Maps and Colouring

One of the most famous problems in graph theory is the problem of colouring a map (such as the world map) in 4 colours such that neighbouring countries have different colours. The countries in the map are faces, and so when we speak of map colouring we usually mean colouring faces of a planar graph. It turns out that this can be expressed as the standard problem of colouring vertices of a planar graph.

Suppose $G = (V, E)$ is a connected planar graph, with F the set of faces. Then its *dual* is a graph $G' = (V', E')$ where $V' = F$, and

$$E' = \{(u', v') |\ u', v' \text{ are on either side of an edge in } G\}$$

Note that if the same face is on both sides of an edge, then we will have a self loop. Supposing G to be a map, you might consider the vertices of G' to be placed at the capitals of the corresponding countries in G, and an edge appears in G' between capitals of countries sharing a border. As you can see, every edge in G' crosses exactly one edge of G, and these edges are indeed called the dual edges. Note that colouring the faces of G is equivalent to colouring the vertices of G'.

It turns out that countries in maps (or vertices in planar graphs) can indeed be coloured using just 4 colours. This is a celebrated theorem, which was proved after a long effort, and the final proof (considered inelegant by some) reduces the problem to colouring some 1936 "difficult" maps, which are coloured using a computer program. We will not study this proof, but will study a less ambitious version.

Theorem 4 *(5-colour theorem) Every planar graph can be coloured using 5 colours.*

Proof: We prove by induction on the number of vertices. The base case is with $v = 3$ and clearly holds. Let G be any graph with $v \geq 3$. We know that $e \leq 3v - 6$. Thus the average degree $2e/v \leq (6v - 12)/v < 6$. Thus G must have a vertex u with degree at most 5. By induction we know that $G - \{u\}$ can be coloured using 5 colours. If u has degree only 4, or if u has neighbours of only 4 colours, we can colour it using the remaining colour. So u must have 5 neighbours p, q, r, s, t, each having a different colour, say with colours 1,2,3,4,5.

Suppose we change the colour of r to 1. If r already has neighbours of colour 1, we change their colour to 3. If they have neighbours with colour 3, we change them to 1 and so on. In this process if the colour of p does not change, then we colour u with 3. So assume that the colour of 1 does change. But then we know that there is a cycle C containing v (uncoloured) and other vertices of colours 1,3, with q of colour 2 inside it.

Now suppose we change the colour of q to 4. We know that this change will not reach s, because s is outside C while q is inside. Thus we now colour u using colour 2. ■

4 Sperner's Lemma

Sperner's lemma intriguingly mixes triangulated graphs, duals, and amazingly, can be used to prove Brouwer's celebrated fixed point theorem.

A *simplicial subdivision* of a triangle T is a partition into triangular *cells* such that every intersection of two cells is either an edge or a corner[1]. We will also use the term *vertex* to denote a corner.

Suppose each vertex v in the subdivision is assigned a label $L(v)$ from the set $\{0, 1, 2\}$. We will say that L is proper if (a) Corners v_0, v_1, v_2 of T have distinct labels, and the label of a vertex on the boundary of the T between v_i, v_j is either $L(v_i)$ or $L(v_j)$. There is no condition on the labels of the vertices inside T.

[1]A corner, edge, and a triangle are examples, respectively of a 0-simplex, 1-simplex, and a 2-simplex, hence the name.

We will extend the notation and use $L(e)$ to denote the set of labels appearing on the vertices of edge e, and $L(f)$ to denote the set of labels on the vertices of face f.

Theorem 5 (Sperner's Lemma) *Every properly labelled simplicial subdivision has a cell with distinct labels for all 3 corners.*

Proof: Let G denote the graph of the simplicial subdivision. G is planar with triangular faces inside. Consider the dual $D(G)$ of G. $D(G)$ has degree 3, except for the outer face f_o which will have a larger degree.

Let G'' be a subgraph of G' such that $D(e)$ appears in G'' iff $L(e) = \{0, 1\}$. Observe first that f_o will have odd degree in G''. This is because as you move from the corner labelled 0 to the corner labelled 1, the number of edges whose endpoints have both 0 and 1 as labels is odd. Only the duals of these edges will appear in G''.

Since we know that G'' has at least one vertex of odd degree, and because the sum of the degrees of a graph is even, there must be an additional odd number of vertices of odd degree corresponding to the inner faces. Since the degree is at most 3, we know that there must be an inner face f with degree either 1 or 3.

Let f have corners p, q, r. Then in G'' f will have degree 3 if and only if $L(p, q) = L(q, r) = L(r, p) = \{0, 1\}$. This is clearly impossible. Thus f must have degree 1 in G''. But this is possible only if $L(f) = \{0, 1, 2\}$. ∎

Brouwer's theorem for two dimensions effectively says that a continuous mapping from a triangular region to itself must have a fixed point, i.e. if f is the mapping then there must exist point p such that $f(p) = p$.

Suppose our triangle T has corner points v^0, v^1, v^2 which we consider as vectors from some origin. Then a point x (associated vector) inside the triangle is a convex combination of v^0, v^1, v^2, i.e. there exist non-negative numbers x_0, x_1, x_2 such that $x = x_0 v^0 + x_1 v^1 + x_2 v^2$ where $x_0 + x_1 + x_2 = 1$. (Exercise). Thus for each point x inside T we can associate "coordinates" (x_0, x_1, x_2).

Let S_i denote the set of points x such that $y_i \leq x_i$, where $y = f(x)$. Since $\sum_i x_i = \sum_i y_i = 1$ each x must belong to some S_i. Consider any simplicial subdivision of T. For each vertex x in the subdivision we select a label $L(x) = i$ only if $x \in S_i$.

We first show that this rule of selecting labels allows the labelling to be proper. The coordinates of v^0 are $(1, 0, 0)$ and hence $f(v^0)_1 \leq v^0_1$. Thus $v^0 \in S_0$, and we may set $L(v^0) = 0$. Similarly $L(v^1) = 1, L(v^2) = 2$. For any point x on the boundary v^0, v^1 of T we will have $x_2 = 0$. If $y = f(x)$ then $y_0 + y_1 + y_2 = 1 = x_0 + x_1$. Hence either $y_0 \leq x_0$ or $y_1 \leq x_1$. Thus we will surely be able to set $L(x) = 0$ or $L(x) = 1$. Similarly for the other boundaries. Thus we can set the labels on the boundary of T properly.

Thus we must find a cell in the subdivision having labels 0,1,2 on its corners.

Suppose now that we make our cells so small that you can assume that f constant on it. Then we know that any point in that cell belongs to S_0, S_1, S_2. If x is a point in the cell and $y = f(x)$, then we have $x_i \leq y_i$. But we know that $\sum_i x_i = 1 = \sum_i y_i$. Thus we must have $x_i = y_i$, i.e. x is a fixed point.

A more formal argument that shows we can find the fixed point in the limit is omitted.[2]

This description is based on West[2], but also see Huang[1].

Exercises

1. Is the butterfly on 4 inputs planar? If not find a suitable subdivision in it. What about the butterfly on 8 inputs?

2. Consider a sketch of the proof a 4 colour theorem for any planar graph G. The proof is by induction over the number of vertices. G must have a vertex v of degree 5 or less. So let us recurse on $G - \{v\}$, and this by the induction hypothesis will give us a 4 colouring. Consider the difficult case in which v has neighbours p, q, r, s, t, going around clockwise and they are coloured 1,2,3,4,2 respectively. Say we now try changing the colour of p to 3 using 1-3 colour exchanges as in the 5 colour theorem. If this works we are done, so let us assume that there is a 1-3 path from p to r. Similarly there must be a $1 - 4$ path from p

[2] Consider a sequence of subdivisions such that the side length of any cell approaches 0. A uniform subdivision which divides T with lines parallel to the sides is good enough for this purpose. Let T_i denote that triangle in the ith subdivision which has labels 0,1,2. Let x^i denote the centroid of T_i. Now the sequence x^i is contained in a closed and bounded set, and hence has a limit point. As the limit is approached, any norm $|x^i - f(x^i)|$ approaches zero, and hence the limit must be a fixed point.

to s. But now we can change the colour of q to 4, and also the colour of t to 3. After that we assign 2 to v.

Find the mistake in the proof.

3. Instead of drawing graphs on the plane or a sphere, suppose we decide to draw them (without intersection) on the surface of a torus. Show that Euler's formula is not valid. Instead, for graphs drawn on the torus we have $v + f - e = 0$, 1, or 2. Give smallest examples that you can think of for each of the 3 possibilities.

   (You may wonder what happens on objects with more holes. The number of holes (defined more formally, of course) is called the genus of an object. For graphs on a surface with genus g, we can have $v - f + e = 2 - 2g$, as it turns out.)

4. Say a planar map is non-degenerate if all vertices have degree 3, i.e. borders of only 3 countries meet at a point. Suppose that in a non-degenerate planar map, all faces have an even number of edges (or all countries have an even number of neighbours). Show that such a map can be coloured in 3 colours.

5. Show that a n node planar graph has treewidth $O(\sqrt{n})$.

6. Suppose a set of identical circular coins are placed on a table. The coins lie flat on the table, but may touch each other. (a) Show that there must exist a coin which touches at most 3 other coins. (b) Consider the intersection graph of the coins, i.e. the graph in which each coin is a vertex and there is an edge between coins if they touch. Show that this graph can be coloured using 4 colours. (Unless specified otherwise, "colouring a graph" means colouring the vertices such that adjacent vertices have different colours).

7. You may have noted that a football is made of pentagonal and hexagonal faces. The graph has degree 3, i.e. at most 3 faces meet in a point. How many pentagons does the football have?

References

[1] J. Huang. On the Sperner Lemma and its applications, 2004. www.cs.cmu.edu/~jch1/research/old/sperner.pdf

[2] D. B. West. *Introduction to graph theory*. Prentice Hall Inc., 2001.