i. A pointer to a concatenable queue $Q[v]$, storing the portion of $U(v)$ not belonging to $U(\text{FATHER}[v])$ (if $v$ is the root, then $Q[v] = U(v)$).

ii. An integer $J[v]$ denoting the position of the left support point on $U(v)$.

This interesting data structure uses only $O(N)$ space, where $N$ is the size of the current point set. Indeed, the skeletal tree $T$ has $N$ leaves and $N - 1$ internal nodes, while the points stored in the concatenable queues represent a partition of the point set.

Since the operations of splitting and splicing concatenable queues are standard, we shall concentrate on the operation BRIDGE for which Overmars and van Leeuwen (1981) propose the following solution.

**Lemma 3.1.** The bridging of two separated convex chains of $N$ points (in total) can be done in $O(\log N)$ steps.

**Proof.** Given two U-hulls $U_1$ and $U_2$ and two vertices $q_1 \in U_1$ and $q_2 \in U_2$, each of these two vertices can be readily classified with respect to the segment $q_1q_2$ as either reflex, or supporting, or concave. (See Section 3.3.6 for an explanation of these terms.) Depending upon this classification there are nine possible cases, which are schematically illustrated in Figure 3.21(a). The wiggly subchains are those which can be eliminated from further contention for containing a support point. All cases are self-explanatory, except the case
(q₁, q₂) = (concave, concave), which is further illustrated in Figure 3.21(b). Let line l₁ contain q₁ and its right neighbor on U₁; similarly let l₂ contain q₂ and its left neighbor on U₂, and let p be the intersection of l₁ and l₂. Recall that U₁ and U₂ are separated, by hypothesis, by a vertical line l. Assume, at first that p is to the right of l. We observe that support point p₁ can only belong to the shaded region, and that u has a lower ordinate than v. This implies that each vertex q'' on the subchain to the right of q₂ appears concave with respect to the segment q'q'', where q' is any vertex of U₁. This shows that the chain to the right of q₂ can be eliminated from contention, but no similar statement can be made for the chain to the left of q₁. If intersection p is to the left of l, then we can show that the chain to the left of q₁ can be eliminated.

In all cases a portion of one or both hulls is eliminated. If this process starts from the roots of both trees representing U₁ and U₂, respectively, since these trees are balanced, BRIDGE(U₁, U₂) will run in time O(log N), where N is as usual the total number of vertices in the two hulls.

With the function BRIDGE at our disposal, we may analyze the dynamic maintenance of the planar convex hull. A typical situation is illustrated in Figure 3.22. Here points are indexed by their order of insertion into the set. In the data structure T, each leaf corresponds to a point, whereas each nonleaf node corresponds to a bridge and is shown with the pair Q[v], J[v]. It is immediate to realize that data structure T describes a free tree on the set of