How to balance the Govardhan

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Preface

Geometry often tends to be taught in high schools as a challenging intellectual obstacle course. Some students take up this challenge and delight in it; many others feel it to be a pointless waste of time and end up hating the subject. This is most unfortunate because Geometry is hardly pointless. It is the earliest substantial example in the educational curriculum of the deductive and intuitive thinking in Science. Developing geometric intuition is valuable also for higher studies, besides being a source of pleasure.

The goal of this booklet is to pursuade students that Geometry is actually relevant to other subjects and real life. There are many real life problems where high school geometry can provide insights and even solutions. The examples given here attempt to connect geometry to real life, to algebra, and even certain parts of physics. While many of these problems will be tackled by scientists and engineers using calculus (and for more complex variations, Calculus will be unavoidable), geometric proofs, because of their pictorial nature could be considered more intuitively obvious.

In addition, it is hoped that the booklet will also amuse.

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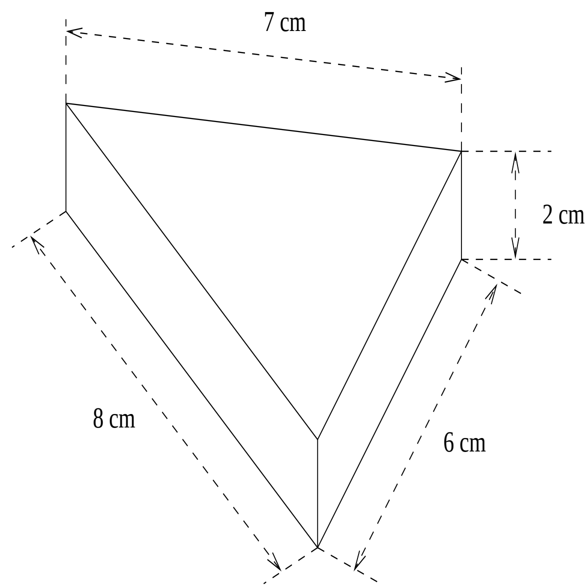
Chapter 1

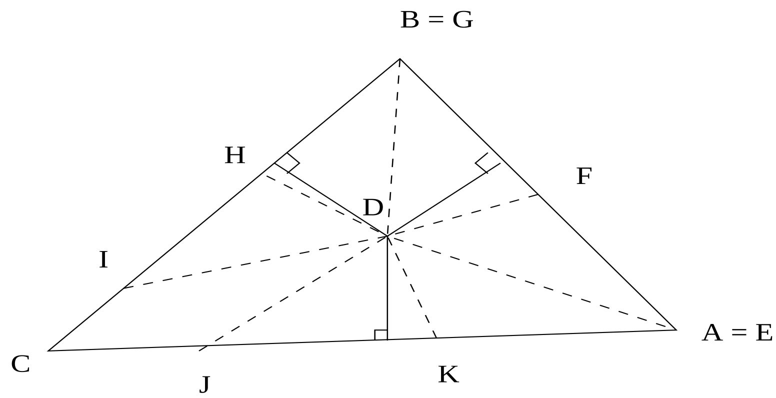
How to cut a cake

Suppose you have triangular cake. It is 2 cm thick, and the sides of the triangle are 6,7,8 cm respectively. The cake has a very thin, but very delicious layer of icing covering its top surface and also the sides. You are asked to divide it amongst 7 people such that each person gets the same volume of cake as well as the same amount (surface area) of the icing.



It turns out that this division can be peformed very easily using a property of the incenter of a triangle. Let ABC denote the triangular face of the cake. Let D be its incenter. Let E,F,G denote the feet of the perpendiculars dropped to the sides from D. Now we know from the definition of the incenter that DE=DF=DG. This property will be very useful in the construction that follows.

First, we divide the perimeter by the number of people who are to receive the cake. In our case, the perimeter is 6+7+8=21, and there are 7 people, so the answer is 3. Now we make marks along the perimeter that they are 3 cm apart. There will be 7 marks, let us call these E,F,G,H,I,J,K. They can be made starting at any point along the perimeter, but for simplicity we will make them starting at A, so we will have A=E, and B=G. Now we simply make a straight cut from D to each mark and this will divide the cake into 7 pieces. We next prove that this division will indeed guarantee equal division of the cake volume as well as the icing area.



First note that each piece got an equal share (3cm) of the triangle perimeter. This proves that everyone gets an equal share (6 sq cm) of the vertical portion of the icing.

Next observe that all pieces except the piece CIDJ are triangular. Considering EF,FG,GH etc. to be the bases of the respective triangle, we see that they all have a common apex, D. The bases of the triangles are all equal (3 cm), and the incenter property assures us that the lengths of the perpendiculars dropped from D are all the same. Thus the areas of these triangles must be all equal. It is left as an exercise for you to show that quadrilateral CIDJ also has the same area as the other triangles. But then, we have proved that the upper surface gets uniformly divided among the pieces. But since the thickness is the same, their volumes must also be equal. Thus we have indeed accomplished equal subdivision of the icing and the cake.

Exercises

1. Prove that area of CIDJ is the same as that of triangle JDK.
2. Devise a strategy to divide a rectangular cake.
3. Suppose the icing is only on the top, and two vertical sides. How will you divide eqaully now?

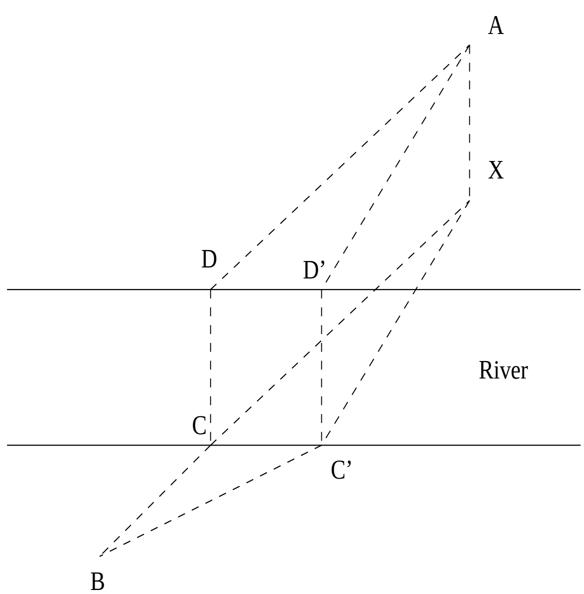
Chapter 2

The Least Cost Road

Suppose you have towns A and B in between which a river flows. As shown in the picture, the river banks are parallel. You are to build a road joining the two towns. A bridge will also be needed; however it is required that the bridge can only be built perpendicular to the river banks. Along the banks the bridge can be anywhere you like. Once the position of the bridge is fixed, then we simply build a straight road connecting the towns to the bridge. So the question is, where should the bridge built, so that the sum of the lengths of the roads is as small as possible.

The theorem that the sum of two sides of a triangle being greater than the third helps us in this.

Here is how to identify the bridge position. First draw a line AX such that AX is perpendicular to the banks, and AX = width of the river. Let C denote the intersection of the bank closer to B with BX. We build the bridge at C, i.e. Draw a line perpendicular to the banks from C and let it intersect the other bank in D. Then CD is the position for the bridge. The roads then are BC,DA.



We now prove that this is the best position for the bridge. Consider any other position, say $C'D'$. Join $C'X$. Note first that CD is parallel to AX, and the two are also equal. Thus $AXCD$ is a parallelogram. Thus $AD = CX$. By similar reasoning, $AD' = C'X$. But then we have

$AD + BC = CX + BC = BX$

But BX is one side of triangle $BC'X$. Thus we have

$BX \leq BC' + C'X = BC' + AD'$

Thus we have shown

$AD + BC \leq BC' + AD'$

Since this applies for any position $C'D'$ for the bridge, we have proved that CD is the best position.

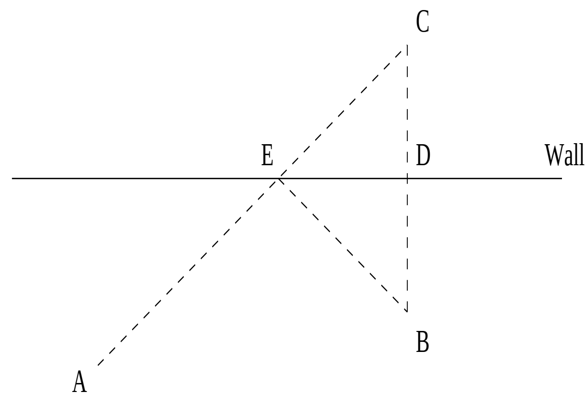
Exercise

Where would you build the bridge if there were two rivers between the town? Assume as before that the river banks are parallel.

Chapter 3

Fermat's Theorem

Here is a game. There are two trees at A,B as shown in the picture, and also a long wall. You are standing at one of the trees, and you are supposed to go to the other tree. However, in between, you should also touch the wall. Where you touch the wall is upto you. But you want to, naturally, ensure that you cover the smallest distance overall.



The theorem that sum of two sides of a triangle is greater than the third helps us again.

Drop a perpendicular from B to the wall, and extend it further to C such that CD=BD, where D is the foot of the perpendicular. Join AC, and let it intersect the wall in E. The path you must follow is AE and then EB. It is left as an exercise for you to show that any other path will be longer. The idea of the proof is very similar to what we saw in the bridge problem.

You can also show that the angle made by AE with the wall is the same as the angle made by EB. What does this remind you of?

It should remind you of optics. If I shine a ray of light on a mirror, it bounces off such that the angle of incidence is equal to the angle of reflection[1].

So this leads us to a surprising fact. Suppose the wall surface was a mirror. Then if we shoot a beam of light from A such that it goes to B after reflecting from the mirror, it would follow the same path that we chose. We know that the path of a light beam is a straight line, which is the shortest path between two points. But what seems to be happening is: if you force light to bounce off a mirror while going from point A to point B, it will do so using the shortest possible path that

[1] The angle of incidence and of reflectio are measured with respect to the perpendicular to the mirror, these angles are equal if and only if the angles made with the mirror by the rays are also equal.

touches the mirror! This observation was first made by the French Scientist Fermat in 1650, and is very important in Optics.[2]

Exercises

1. Show that AE followed by EB is indeed the shortest path to go from A to B while touching the wall.

2. Show that the angle made by AE with the wall as the same as that made by EB.

3. Suppose now that A and B are inside a rectangle. You are to go from A to B, but while touching at least two sides. How will you go such that your path is as short as possible?

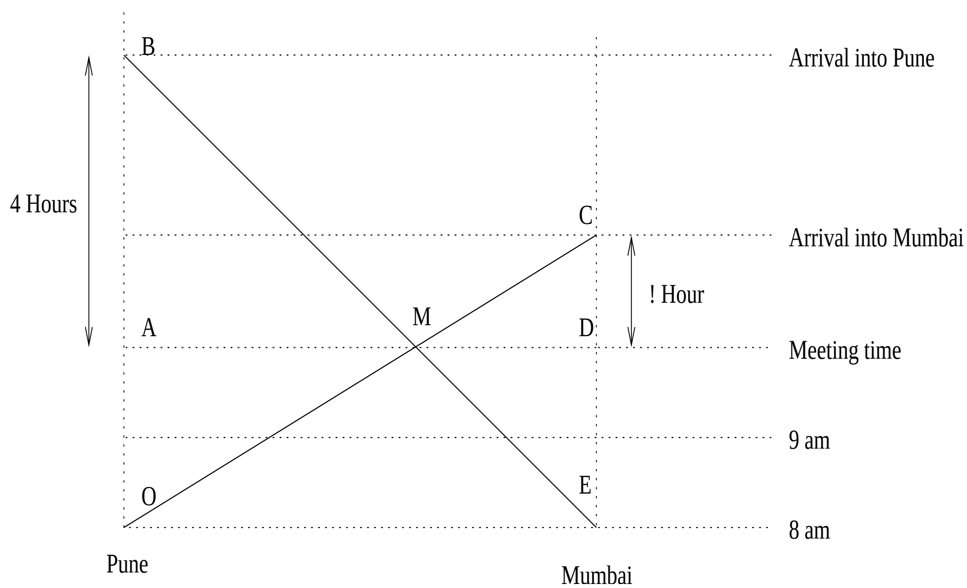
[2]Fermat's observation were more general, and they apply in the case of reflection, as well as refraction. The exact details will take us far from our main topic.

Chapter 4

Geometric Visualization

Suppose a car starts travelling from Pune to Mumbai at 8am. It has some constant but unknown speed. Simultaneously another car start travelling from Mumbai to Pune, also at a constant (but possibly different) speed. At some point, they meet somewhere along the road, but continue without stopping. One hour after the meeting, the car travelling from Pune reaches Mumbai. Whereas, 4 hours after the meeting, the car travelling from Mumbai reaches Pune. How long did each car take for the entire journey?

At first glance, it may not seem that this problem has much to do with geometry. However, we can often visualize a phenomenon using Geomtric objects, and the visualization helps us in solving problems. The basic idea is to represent the given information using a graph, as shown below.



For those unfamiliar with graphs, this picture is a representation of the events that happen between Pune and Mumbai, during the time in which the cars travel. On the vertical line marked "Pune", we will mark events that happen in Pune. While on the vertical line marked Mumbai, we will represent the events that happen in Mumbai. If an event happens in between, say one third of the distance from Pune, then it will be represented on a line drawn at one third the distance between the Pune and Mumbai lines. The horizontal line marked "8am" is used for denoting the events that happen at 8am. Thus an event which happen at 8am in Pune will be shown at the intersection of the 8am line (horizontal) and the Pune line (vertical). From the information given in the problem, we know that this intersection point, marked O, corresponds to the event of the car starting its journey from Pune to Mumbai. Likewise, the point E represents the event of the car from Mumbai starting its journey. Other horizontal lines correspond to other times, e.g. the line marked 9am. As before the intersection of this line with a vertical line for Pune/Mumbai can be used to denote an event happening at 9 am at the corresponding city. At 9 am, no noteworthy event happens in any city, and hence we have left these intersections unlabelled. As before, the events happening at 8:30 will be marked on a horizontal line midway between the 8am and 9am lines. In addition, we may also represent events happening later, say the events at 10 am will be represented on a horizontal line which is as much above the 9 am line as the 8am line is below.

Since the car from Pune eventually reaches Mumbai, its arrival will correspond to a point on the Mumbai line. In the picture we have shown it by the point C. Here we arbitrarily assumed that the arrival will be after 9am, and so C is drawn above the 9am line. A key question you should try to answer is: how would you represent the event of the car passing the halfway point between Pune and Mumbai? This event will have to be on the line (call it v) midway between the Pune and Mumbai lines, since it takes place at half the distance between the cities. It will also have to be on the line (call it h) halfway between the 8am line and the horizontal line through C. From this, you should be able to see that this event will correspond to exactly the midpoint of OC, which is at the intersections of the lines v and h. Proceeding in this way, you will see that every point on line OC can be associated with the travel of the car in the following way. Suppose the car passes a point on the road at a distance d from Pune at some time t. Then the intersection of the vertical line associated with this point on the road and the horizontal line corresponding to time t will lie on the line OC.

In a similar manner, we have chosen to represent the arrival in Pune of the car from Mumbai. By similar reasoning the journey of the car from Mumbai to Pune will be represented by the line EB.

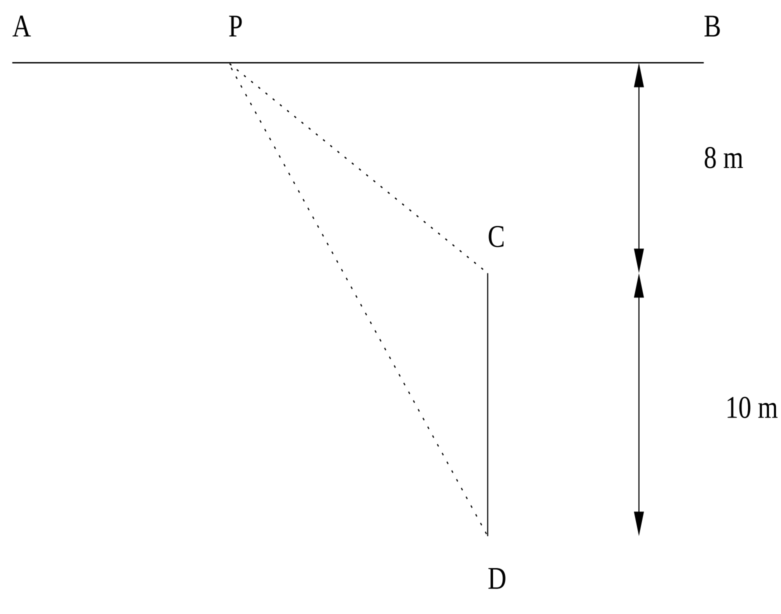
But what can we say about the intersection point M of the lines EB and OC? This point corresponds to the event of the cars meeting. The horizontal line through M gives the time at which the cars meet, whereas the vertical line gives the distance at which they meet. To complete the story, we are told that the arrival of the car into Mumbai happens an hour after the meeting. Thus the vertical distance between the meeting time and the Pune arrival corresponds to one hour. Thus the distance CD corresponds to 1 hour. In a similar manner we note that the distance AB corresponds to 4 hours. Finally, we need to know the total time for each journey, so it suffices if we compute what vertical distance ED is.

Now we can compute the length of ED in terms of other lengths using similar triangles. Note that triangles EMD and BMA are similar. Thus it follows that ED/BA=DM/AM. But triangles CMD and OMA are also similar. Thus CD/OA=DM/AM. Thus it follows that $CD/OA = ED/BA$. But noting that $OA = ED$, we have $ED^2 = CD \cdot BA$. We have been saying so far that CD corresponds to 1 hour – but we could choose it as our unit length, in which case we will have $CD = 1$, and $BA = 4$. Thus we will have $ED^2 = 4$, or $ED = 2$. Thus the journey from Mumbai to Pune takes 6 hours, while the other takes 3. Done!

Chapter 5

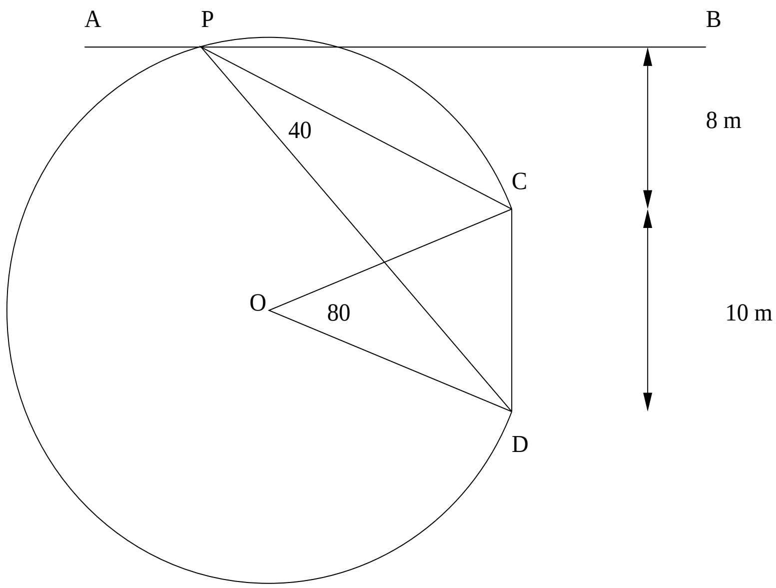
The best vantage point

In the picture below, AB is a road, and CD is a roadside advertisement. The advertisement happens to have a very nice picture, which you want to photograph. Your goal is to determine the best place on the road from where to take the picture (assume that for some reason you cannot get off the road). You want the picture to appear as large as possible in your photograph. In other words, the point P from where you take the picture must be such that $\angle CPD$ is as large as possible. How will you find this point P?



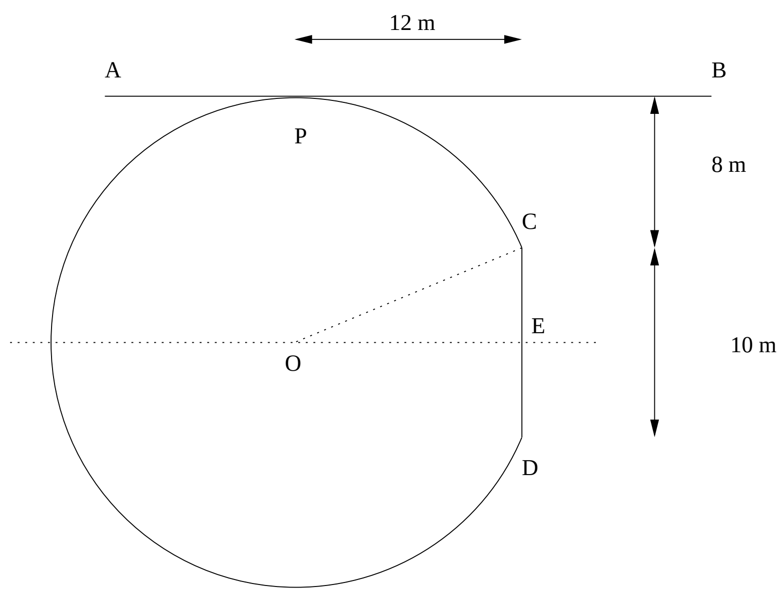
Let us think a little about how we might go about solving this puzzle. To get started, you might fix P arbitrarily, and measure the angle CPD. Suppose this turns out to be 39 degrees. You may then wonder, could it be greater than this? Is it possible, for example that we can find a point P such that the angle is 40 degrees?

This is a standard exercise from high school geometry. The key idea is the theorem which says: The angle subtended by a chord at the center of a circle is twice the angle inscribed in the arc on top of the chord. So if we draw a circle with center O such that CD is its chord, and such that $\angle COD = 80°$, then P must lie on the circle. But since P must also lie on road, then we know that it must be one of the points in which the circle and the road intersect.



Now you might want to ask, can we get an angle of 41°? But then you must also ask: will its circle be larger or smaller? Clearly, the smaller the circle, the bigger the angle CD will subtend at its center, and hence the bigger the inscribed angle will be.

So we want the circle to be as small as possible and yet intersect the road. As the circle shrinks, the points in which it intersects will get closer to each other; eventually they merge into a single point when the circle is tangential to the road. If you shrink the circle further, then it will not intersect the road any longer. Thus the largest circle that touches the road will have to be tangential to the road, and also as we know, have CD as its chord.



We now show how to find this circle. If CD is to be a chord, then its center must lie on the perpendicular bisector of line CD. Now the center of this circle O must be at the same distance from the road as it is from points C and D. Since the perpendicular bisector is parallel to the road and at a distance 13 meters from the road, we have $OC = 13$. But then the OED is a right angled triangle with hypotenuse and one side of length 13 and 5 respectively, so the other side OE must be of length 12.

Exercise

What if the advertisement is at an angle to the road, and also away from it?

Chapter 6

How to balance the Govardhan

You have all read the story of Krishna balancing the Govardhan mountain on the tip of his little finger. Doing this clearly requires great strength, but it is also not clear how Krishna would have decided where to place the mountain on his finger so that it would balance. In this chapter we are going to see how to balance triangles on one finger. We will show that a triangle will balance if you place your finger under its centroid. Some of the arguments we will use will also give you a clue about how to balance the Govardhan. In any case, you will see that balancing the Govardhan requires not only strength, but also cleverness!

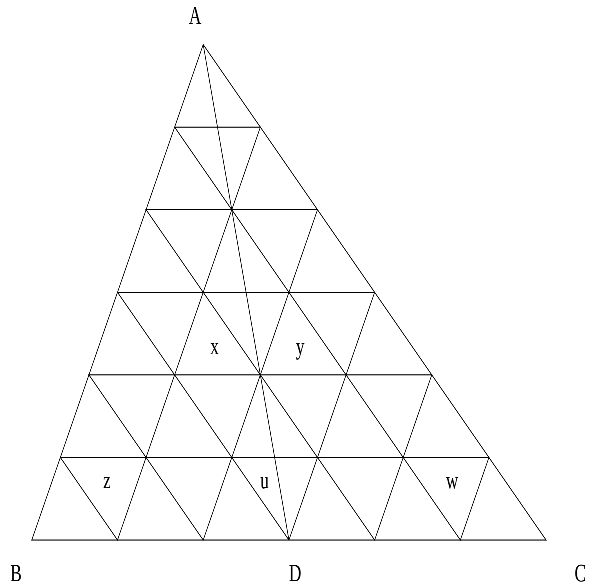
The simplest mechanism where we worry about balancing is a see-saw. If two children of equal weight sit at equal distance from the pivot of the see-saw, the see-saw can remain horizontal. You probably have observed that if two children sit on one side and one child on the other (all of equal weight), then the two children must sit at half the distance from the pivot as compared to the single child. You might also have played a game in which two children seated on the seesaw start moving towards each other – can you guess how they need to move so that the see-saw remains balanced? You will note that if the children have the same weight, then they must move an equal distance towards (or away from) each other in order to keep the seesaw balanced.

Surprisingly, these simple observations help us in balancing a triangle, say one cut out of card-board, and having possibly unequal sides. We will prove that it can be balanced by placing its centroid on the pivot.

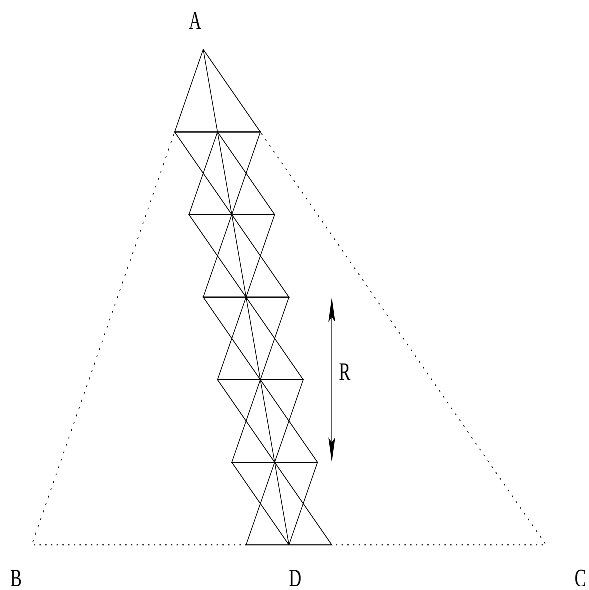
We begin with some principles from physics which we will state without proof. The first principle is very similar to the seesaw game mentioned above. Suppose we have a certain object that balances when a point P on the object is placed on a pivot. Now suppose parts of the object of equal weight move an equal distance towards each other. Then the balance will not be upset. We will call such movements *balance preserving movements.* The second principle is even more self evident: suppose that the entire weight of an object can be moved inside a circle (possibly much smaller than the original object), then the pivot must have been under that circle.

To apply the principle to the triangle, suppose that we have somehow managed to balance it by placing some point P of it on a pivot. We will argye that it is possible to apply balance preserving movements such that all weight of the triangle shifts inside a circle around the centroid. Furthermore, we will argue that we can make this circle as small as we wish, thereby establishing that the pivot must be inside the intersection of all these circles. Since it will turn out that the only point common to all possible circles must be the centroid, then will follow that the pivot must be under the centroid.

Let AD be the median of the triangle. Suppose we divide the triangle into n^2 smaller triangles, by dividing each side into n parts and slicing parallel to the sides. This is shown below for $n = 6$.



Now consider triangles such as z and w at equal horizontal distance from the median. We move these an equal distance horizontally so that they overlap with one another over the triangle u. Similarly triangles x, y which are likewise at equal distance from the median (horizontally) are also moved horizontally. In this case each is moved by distance equal to half the base, so that the medians of both bases will appear on the median. All such moves are balance preserving. After making all such moves, we get the picture shown below. Note that many triangles in this picture represent a stack of triangles. For example, the triangle with its base at the bottom is a stack of 6 triangles.



Next we start moving the small triangles towards each other, always staying on the median, and in balance preserving moves. Our goal is to get the triangles moved into as compact a region as possible.

How compact can the region become? I claim that we should be able to use balance preserving moves to get all triangles into a region two strips tall, like the region R shown. To do this we simply keep moving triangles at the extreme ends towards each other. Suppose the stacks at the extreme endpoints contain I, J triangles, with $I \leq J$ for example. Then we move I triangles from each stack 1 step towards each other. As a result, one of the stacks must vanish. We repeat the procedure, and this repetition can be done until triangles are all in just 2 adjacent strips. This region is clearly contained in a circle of radius at most as large as the length of the largest side of the (small) triangle. So if L denotes the length of the largest side of triangle ABC, then the radius of the circle is L/n.

The key points to note now are that the circle must always intersect the median AD, no matter what n we use. But by increasing n we can shrink the circle as much as we want. Now the only point common to all these circles must be a point on the median AD, since we know that the circle must lie on AD. Thus the pivot must be somewhere below AD.

But now, note that this argument could be applied to the other medians as well. Thus the pivot must lie under the other medians as well. But the only point which is on all medians is the centroid, and hence the pivot must be under the centroid. Thus proved!

6.1 General shapes

How can you determine the pivoting point for an arbitrary shape? This is easy if it can be divided into triangles. Then just as above we move triangles together into as compact a region as possible. If we can make this region arbitrarily small then we can find the balancing point as we did above.

What if the given region (e.g. circle, or a map of India) cannot be divided into triangles exactly? Turns out that any figure can be divided into regions some of which are triangles and others are not, but such that the area of the non-triangular regions can be made as small as we wish by choosing a larger number of triangles. This is shown below for a circle. As you can see, as the number of triangles increases, more and more area in the circle is covered. What is shown is not the only way to do this, of course. It is customary to state this as: any figure can be approximated to any degree of accuracy by choosing a large enough number of triangles. So this gives us a procedure for computing the pivoting point approximately: find the pivoting point for the collection of triangles chosen above. As the accuracy of our collection increases, so does the accuracy of our pivoting point.

We can extend this idea for solid shapes (e.g. Govardhan) as well: we first break the shape into layers, accumulate the weight of each layer into a very small region, then accumulate those small regions using balance preserving movements. The pivot must be below the final region (as before we must argue that we should be able to shrink it as much as we want).

There is a catch in all this. It may be possible to move the weight to a small region, but that region is not a part of the object which we started with. This would happen for example, if we have a triangle with a hole around the centroid. In this case, there is no way to balance the triangle (or other such objects) on a single point.

It should be noted that all such calculations are typically done by engineers using Calculus, but the underlying principle is what we have discussed.

Exercise

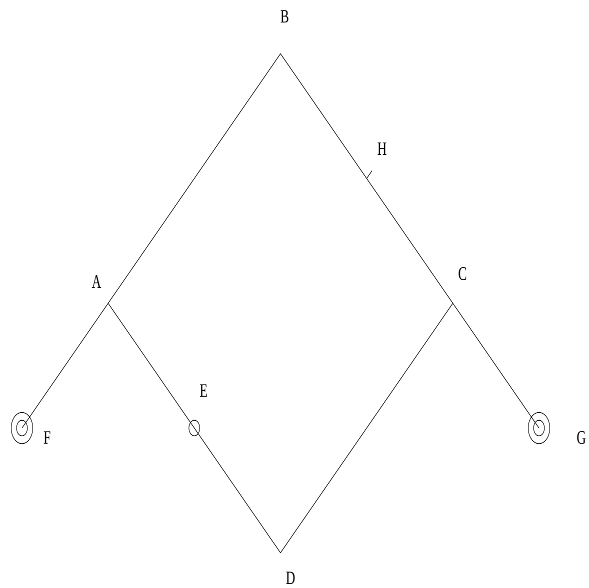
1. Consider 2×2 square from which one 1×1 square has been cut off. Can you give a compass and ruler construction to determine where it will balance? Actually do this and check your answer.

2. It is in fact possible to find the balancing point without any calculations or any mathematics even. Begin by balancing the object, say a triangle on 3 fingers. Then gradually try to move your fingers together, without jerky motion, and without touching the object with anything else. Turns out that your fingers will close around the centroid. Ask your physics teacher for the explanation!

Chapter 7

Pantograph

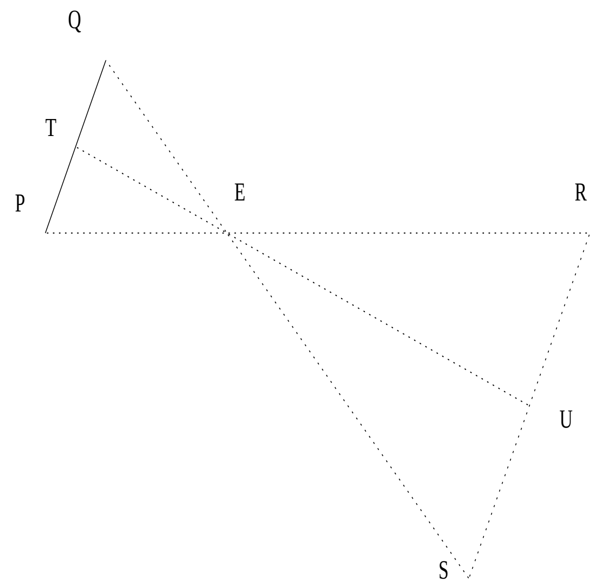
Given below is a schematic picture of a device which can help you make an enlarged copy of any drawing. This device is called a pantograph, and was invented by Christoph Scheiner in 1603.



In this picture FB, BG, AD, DC represent strips of wood or metal. At F, G there are pens. At point E, there is a vertical pin using which the apparatus can be attached to the table. However, after the attachment, the strip AD is free to rotate around E. The joints at A, B, C, D are hinge joints, so that the angles between the strips can change if needed. The lengths of the strips are important, and we have $AB = BC = CD = DA = DG$, and $AF = AE = ED$. Suppose now that I make a drawing using the pen at F. What figure is drawn because of the pen at G? Turns out that the same drawing, but of twice the size, will be made by the pen at G.

To prove this let us begin by using H to denote the midpoint of BC. Clearly, $EH = HG$, and further $\angle FAE = \angle EHG$. Thus, triangles FAE, EHG are isoceles and similar, with $FA = 2 \cdot EH$. So $\angle AFE = \angle HEG$, and AB, EH are parallel, we know that F, E, H must be collinear. So we have established that no matter how the pens move, pen F, G are always on the opposite sides of E, and $FE = 2 \cdot EG$. Let us call this the basic pantograph property.

Suppose now that we draw the straight line PQ using the pen F. We will show that during this movement the pen G traces a straight line RS and further $RS = 2 \cdot PQ$.



The point E shown in this figure is the same as that in the previous, i.e. it is the point where the apparatus is attached to the table. Now join PE and extend it so that $ER = 2 \cdot PE$ as shown. So using the basic pantograph property, we know that when the pen F was at point F, the pen G must be at point R. Likewise, join QE and extend it so that $ES = 2 \cdot QE$. In a similar manner we know that when the pen F is at Q, the pen G must be at S. Let T be any point on PQ. As before, join TE and extend it so that $EU = 2 \cdot TE$. We know as before that when the pen F is on T, the pen G must be on U. If we can now argue that U must be on line RS, then we are done, because this applies to all points U traced out by pen G.

For the proof, note first that triangles PET, REU must be and similar. This is because $RE = 2 \cdot PE$, $EU = 2 \cdot ET$, and $\angle PET = \angle REU$. Thus it follows that $\angle PTE = \angle RUE$. Similarly we can prove that triangles QET, SEU must similar, and also that $\angle QTE = \angle SUE$. Thus $\angle RUE + \angle SUE = \angle PTE + \angle QTE$. But $\angle PTE + \angle QTE = 180°$. Thus $\angle RUE + \angle SUE = 180°$, from which it follows that R, U, S must lie on a straight line.

So we have proved that G draws straight lines of twice the size as those drawn by F. Suppose now that pen F draws a triangle – then by applying the same logic to each line, we can conclude that pen G must also draw a triangle of twice the side lengths. In a similar manner we can argue (see the exercises) that other figures drawn by G will also be twice the size of the figures drawn by F.

Exercise

1. Show that if F draws a circle then G draws a circle of twice the size.

2. Design a pantograph in which one pen draws figures of thrice the size drawn by the other pen.

3. What we have shown is only one way to attach metal strips so that we get a pantograph. Device other ways.

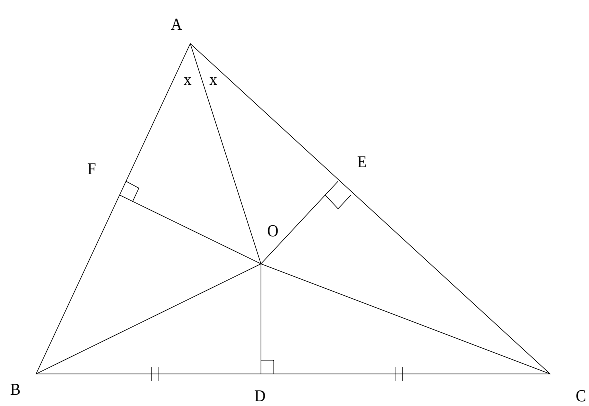
Chapter 8

All triangles are isosceles

Suppose I claim that the medians of a triangle meet in a point. How can we check this claim? One way is to draw many triangles and their medians, and check if they do indeed meet in a point. Such a method can verify that all the triangles *we have seen* do in fact have coincident medians; it however does not really tell us whether we should expect the next triangle that we will draw will have coincident medians or not. Because of this, the notion of a mathematical proof was invented. If we believe in logic, and if we believe in Euclid's postulates, then if we are given a proof of the coincidence of the medians of a triangle, we believe that to be true for all triangles, including the ones we have never seen before.

Discovering and writing proofs can be tricky. If we are not careful, it is possible to draw wrong conclusions. An argument which appears to be logically correct and which leads to a wrong conclusion is termed a *fallacy*. Fallacies are entertaining, but they are also educative because they warn us of pitfalls in writing proofs. Here is an example of a fallacy. See if you can spot the error in it.

Claim: All triangles are isosceles.
"Proof:" Consider any triangle ABC. Let the bisector of $\angle A$ meet the perpendicular bisector of BC in point O. Let OD, OE, OF be the perpendiculars to BC, AC, AB respectively.



Note first that in triangles OEA, OFA we have $\angle OAE = \angle OAF$, base OA is common, and $\angle OEA = \angle OFA = 90°$. Thus these triangles are congruent. Thus $OF = OE$.

Note also that in triangles OBD, OCD, $BD = CD$, OD is common, and $\angle ODB = \angle ODC = 90°$. Thus the triangles are congruent. Thus $OB = OC$. Also, $\angle OBD = \angle OCD$.

But then in triangles OBF, OCE we have $OF = OE$ (proved above), $OB = OC$ (proved above), and $\angle OFB = \angle OEC = 90°$. Thus these triangles are also congruent. Thus $\angle OBF = \angle OCE$.

But $\angle ABC = \angle OBF + \angle OBD$. But since $\angle OBD = \angle OCD$ and $\angle OBF = \angle OCE$ we get $\angle ABC = \angle OCD + \angle OCE$. But the right hand side of this is simply $\angle ACB$. Thus $\angle ABC = \angle ACB$. Thus the triangles are isosceles. ∎

Can you spot the error? I will not give the solution away. However here is a hint: draw the figure accurately, and then you will see what has gone wrong.

Exercise

1. Puzzle in which rectangle is cut and reassembled.
2. fallacy involving multiplication by zero.
3. fallacy involving square roots.