Binary Search and Variants

- Applicable whenever it is possible to reduce the search space by half using one query

- Search space size N
  number of queries = $O(\log N)$

- Ubiquitous in algorithm design. Almost every complex enough algorithm will have binary search somewhere.

Classic example

- Given a sorted array A of integers, find the location of a target number x (or say that it is not present)

Pseudocode:

```
Initialize start ← 0, end ← n;
Locate(x, start, end){
    if (end < start) return not found;
    mid←(start+end)/2;
    if (A[mid] = x) return mid;
    if (A[mid] < x) return Locate(x, mid+1, end);
    if (A[mid] > x) return Locate(x, start, mid-1);
}
```

Other examples

- Looking for a word in the dictionary
- Debugging a linear piece of code
- Cooking rice
- Science/Engineering: Finding the right value of any resource
  - length of youtube ads
  - pricing of a service

Egg drop problem

- In a building with n floors,
  find the highest floor from where egg can be dropped without breaking.
- $O(\log n)$ egg drops are sufficient.
- Binary search for the answer.
- Drop an egg from floor x
  - if the egg breaks, answer is less than x
  - if the egg doesn't break, the answer is at least x
- Using the standard binary search idea, start with $x=n/2$.
  If egg breaks, then go to $x=n/4$ and it doesn't then go to $x=3n/4$.
  And so on

Egg drop: unknown range

- In a building with *infinite* floors,
  find the highest floor h from where egg can be dropped without breaking.
- *O(log h)* egg drops sufficient?
- Exponential search: Try floors, 1, 2, 4, …, till the egg breaks.
- The egg will beak at floor 2^{k+1}, where
  $2^k \leq h < 2^{k+1}$
- Then binary search in the range $[2^k , 2^{k+1}]$
- Total number of egg drops $\leq k+1+\mathrm{k} = 2k+1 \leq 2 \, log \, h +1$.
- Is there a better way?

Lower bound

- Search space size: N
  Are $log\ N$ queries necessary?

- Yes. When each query is a yes/no type, then the search space gets divided into two parts with each query (some solutions correspond to yes and others to no).

- One of the parts will be at least $N/2$.

- In worst case, with each query, we get the larger of the two parts.

- To reduce the search space size to 1, we need $log\ N$ queries

Lower bound

- Search space size: N
  Are $\log N$ queries necessary?
- Another argument based on information theory.
- Each yes/no query gives us 1 bit of information.
- The final answer is a number between 1 and N, and thus, requires $\log N$ bits of information.
- Hence, $\log N$ queries are necessary.
- Ignore this argument if it is hard to digest.

Exercise: subarray sum

- Given an array with n positive integers,
  and a number S,
  find the minimum length subarray whose sum is at least S?

- Subarray is a contiguous subset, i.e.,
  A[i], A[i+1], A[i+2], …, A[j-1], A[j]

- [10, 12, 4, 9, 3, 7, 14, 8, 2, 11, 6]

  S = 27

- Can we do this in *O(n log n) time*?

Exercise: subarray sum

O(n^2) algorithm:

for ($l \leftarrow$ 1 to n){

T $\leftarrow$ sum of first l numbers.

for ($j \leftarrow$ 0 to n-l-1){

if T $\geq$ S return l and the subarray (j, j+l-1);

T $\leftarrow$ T - $A[j]$ + $A[l+j]$;

}

}

- Two for loops one inside the other. Each makes at most n iterations. Hence, O(n^2) time.

Exercise: subarray sum

O($n \log n$) algorithm. Approach 1:

Binary search for the minimum length l.

For the current value of l:

> check if there is a subarray of length l with sum at least S.
> This check can be done in $O(n)$ time. See the inner loop on previous slide.

Exercise: subarray sum

- O($n \log n$) algorithm. Approach 2 (suggested by students):

- First compute all the prefix sums and store in an array.
  $O(n)$ time.

- prefix_sum[0] ← A[0];
  for (i ← 1 to n-1)
  prefix_sum[i] = prefix_sum[i-1]+A[i];

- Any subarray sum from i to j can now be computed in O(1) time as prefix_sum[j] - prefix_sum[i-1].

- Now, for each choice of starting point, do a binary search for the minimum end point such that the subarray sum is at least S.

- If sum of a subarray < S, then choose a larger end point, otherwise smaller.

Exercises

- Given two sorted arrays of size n, find the median of the union of the two arrays.
  $O(\log n)$ time?

- Given a convex function $f(x)$ oracle, find an integer x which minimizes $f(x)$.

- Land redistribution:
  given list of landholdings $a_1, a_2, \ldots, a_n$,
  given a floor value f,
  find the right ceiling value c

Division algorithm

- As we find the next digit of the quotient,
  the search space of the quotient goes down by a factor of 10.

- This could be called a denary search.

- For binary representation of numbers, the division algorithm will be a binary search.

Finding square root

- Given an integer a, find $\sqrt{a}$
- Start with a guess x
- If $x^2 > a$, then the answer is less than x
- Else the answer is at least x.
- This way we can do binary search for $\sqrt{a}$
- Additional exercise: find a division like algorithm.

Finding square root

- Given an integer a, find $\sqrt{a}$ up to l digits after decimal.
- Can we compute it in $O(l)$ iterations?
- Yes.

More applications

- Finding inverse of an increasing function?
- Finding root of polynomial?
- Finding the smallest prime dividing N ?
- Is sorting a kind of binary search?
  $O(n \log n)$ comparisons necessary?

More applications

- Finding inverse of an increasing function?
- For any given x, we have a method to compute $f(x)$. For a given y, we want to compute $f^{-1}(y)$.
- Make a guess x and compare it $f(x)$ with y
- If $f(x) < y$ then the answer is larger than x
- Else the answer is at least x.

More applications

- Finding a root of polynomial $f(x)$?
- Always maintain two points a and b such that $f(a) > 0$ and $f(b) < 0$.
- To find the starting points one can do exponential search.
- Check whether $f((a+b)/2) > 0$.
- If yes, then there is a root between $(a+b)/2$ and b.
- Else, there is a root between $(a+b)/2$ and a.

More applications

- Finding the smallest prime dividing N ?
- No.
- We can make a guess x. If x does not divide N, then we cannot say anything about where should be the smallest prime dividing N.

More applications

- Is sorting a kind of binary search?
  $n \log n$ comparisons necessary?
- Yes.
- When we have not made any comparisons, then any of the $n!$ rearrangements is a possible answer.
- So the search space size is $n!$.
- Each comparison will reduce the search space size by only 1/2 (in worst case).
- Hence, $\log (n!) \geq n \log n - n \log e$ comparisons are necessary.

Analyzing algorithms

- Comparison between various candidate algorithms
- Why not implement and test?
  - too many algorithms
  - depends on input size, how inputs are chosen
- Will count the number of basic operations like addition, comparison etc.
- And see how this number grows as a function of the input size. This measure is independent of the choice of the machine.

$O(\cdot)$ notation

- For input size n, running time $f(n)$

- We say $f(n)$ is $O(T(n))$
  if for all large enough n and some constant c,
  $f(n) \le c \cdot T(n)$

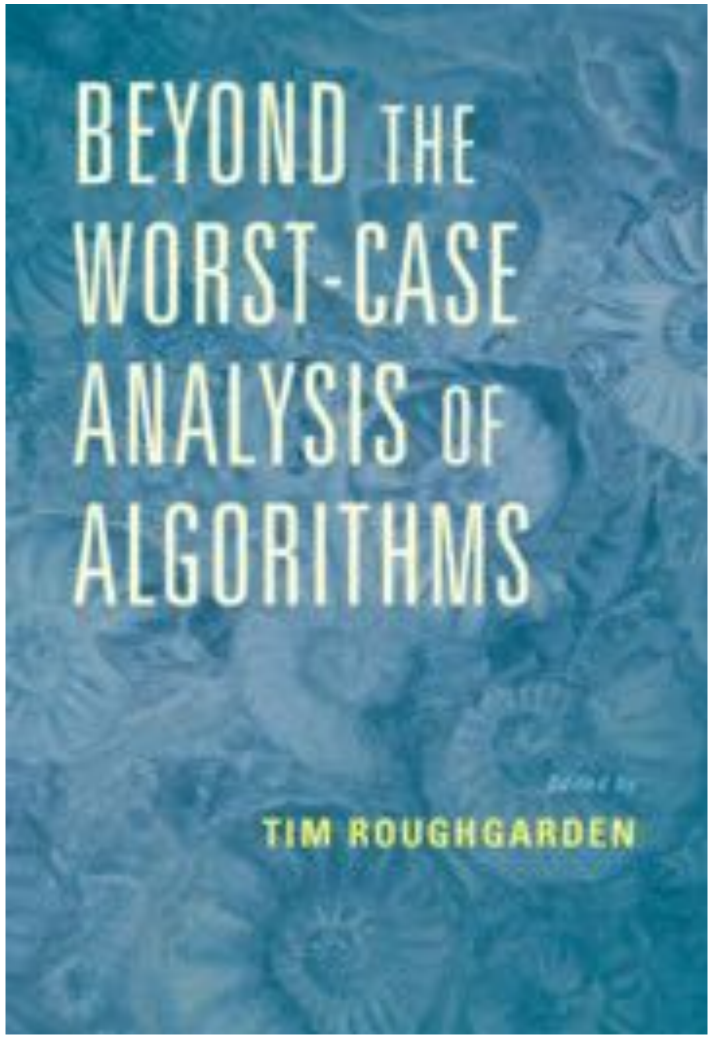
$O(\cdot)$ notation

- Why do we ignore constant factors?
- Because it's not possible to find the precise constant factor. Various basic operations do not take the same amount of time.
- Is $O(n)$ always better than $O(n \log n)$?
- For large enough inputs, yes. But, depending on the hidden constant factors, it's possible that $O(n \log n)$ algorithm is faster on reasonable size inputs.

Worst case analysis

- Worst case bound: running time guarantee for all possible inputs of a size.
- There could be algorithms which are slow on a few pathological instances, but otherwise quite fast.
- Why not analyze only for “real world inputs”?
- It’s not clear how to model “real world inputs”.
- For many algorithms, we are able to give worst case bounds.

Worst case analysis



Out of scope of this course

Describing algorithms

- Find the maximum sum subarray of length k

s = 0;
for (i=0, $i < k$, i++) s=s+A[i];
m = s;
for (i=0, $i < n$-k, i++){
    s = s - A[i] + A[k+i];
    if (s > m) m = s;
}
return m;
	Compute the sum of first k numbers. We will go over all length k subarrays from left to right. In an iteration, update the sum by subtracting the first number of the current array and adding the number following the last one. If the sum is larger than the maximum seen so far, we update the maximum.	s ← sum of first k numbers;
		for (i ← 0 to n-k-1){
		    s ← s - A[i] + A[k+i];
		    m ← max(m, s);
		}
		return m;

Precise, but hard to understand. Error prone.

Not precise. Open to multiple interpretations.

Somewhere in the middle.

Describing algorithms

- Two sorted (increasing) arrays A and B of length n. Count pairs (a,b) such that $a \in A$ and $b \in B$ and $a < b$

```
j=0; count = 0;
for (i=0, i < n, i++){
    while (A[i] >= B[j]) j++;
    count=count + n - j;
}
return count;
```

j $\leftarrow$ 0; count $\leftarrow$ 0;
for (i $\leftarrow$ 0 to n-1){
    keep increasing j till we get $B[j] > A[i]$;
    count $\leftarrow$ count + n - j;
}
return count;

O(·) notation

- True or False?
- $2n+3$ is $O(n^2)$
  - True
- $1^2 + 2^2 + \ldots. + (n-1)^2 + n^2$ is $O(n^2)$
  - False (it is $\Theta(n^3)$)
- $1 + 1/2 + 1/3 + \ldots + 1/n$ is $O(\log n)$
  - True
- n^n is $O(2^n)$
  - False

$O(\cdot)$ notation

- True or False?
- 2^{3n} is $O(2^n)$
  - False
- $(n+1)^3$ is $O(n^3)$
  - True
- $(n+\sqrt{n})^2$ is $O(n^2)$
  - True
- $log(n^3)$ is $O(log\ n)$
  - True

Principles of algorithm design

First principle: reducing to a subproblem

- Subproblem: same problem on a smaller input
- Assume that you have already built the solution for the subproblem and using that try to build the solution for the original problem.
- Subproblem will be solved using the same strategy.
- Implementation: recursive or iterative

First principle: reducing to a subproblem

- Example 1: finding minimum value in an array.
- Suppose we have already found minimum among first *n-1* numbers, say min_{n-1}
- $min_n = \text{minimum}(min_{n-1}, A[n])$
- Iterative implementation:
  Go over the array from *1* to *n* and maintain a variable *min*
- Invariant: after seeing *i* numbers, *min* will be the minimum among first *i* numbers.
- $min_i = \text{minimum}(min_{i-1}, A[i])$

First principle: reducing to a subproblem

- $min_i = \text{minimum}(\, min_{i-1}, A[i]\,)$

- Iterative implementation

$min \leftarrow A[1];$

for ($i \leftarrow 2$ to n)

    $min \leftarrow \text{minimum}(min, A[i])$

- Recursive implementation

f(A, i):

    if $i=1$ return $A[1]$;

    else return minimum($f(A, i\text{-}1)$, $A[i]$);

Compute $f(A, n)$;

Maximum subarray sum

- Given an integer array with positive/negative numbers. Find the subarray with maximum possible sum.

- 1, 2, -5, 4, -6, $\boxed{8, 7, -3, 2, 10, 3,}$ -7, 4, 2
- $O(n^2)$ algorithm
- For every choice of starting point, go over all choices of end points and maintain the sum between starting and end points.

- Maintain the *max_sum* value by comparing with the current sum

Maximum subarray sum

O(n^2) algorithm:

```
max_sum ← 0;
for (start ← 1 to n){
    curr_sum ← 0;
    for (end ← start to n){
        curr_sum ← curr_sum + A[end];  // update the current sum
        max_sum ← maximum(curr_sum, max_sum);
    }
}
```

Max subarray sum: subproblem

- Suppose we have already found the maximum subarray sum for $A[1 \ldots n\text{-}1]$, say max_sum_{n-1}
- How do we compute max_sum_n
- Two kinds of subarrays of A:
  1. subarrays of $A[1 \ldots n\text{-}1]$
  2. subarrays of A which end at n
- max_sum_n = Maximum(max_sum_{n-1} ,

$$\left.\begin{array}{l} Sum(1 \ldots n), \\ Sum(2 \ldots n), \\ \vdots \\ Sum(n \ldots n), \;) \end{array}\right\} O(n)$$

$$T(n) = T(n\text{-}1) + O(n) \qquad \Rightarrow T(n) = O(n^2)$$

Max subarray sum: subproblem

- Improvement ?
- Observation:

$$Sum(1 \ldots n) = Sum(1 \ldots n\text{-}1) + A[n],$$
$$Sum(2 \ldots n) = Sum(2 \ldots n\text{-}1) + A[n],$$
$$\vdots$$
$$Sum(n\text{-}1 \ldots n) = Sum(n\text{-}1 \ldots n\text{-}1) + A[n],$$

- Maximum($Sum(1 \ldots n), Sum(2 \ldots n), \ldots, Sum(n\text{-}1 \ldots n)$)

  =

  Maximum($Sum(1 \ldots n\text{-}1), Sum(2 \ldots n\text{-}1), \ldots, Sum(n\text{-}1 \ldots n\text{-}1)$) + $A[n]$

- We have converted it to another problem on first *n-1* numbers

Max subarray sum: subproblem

- Improvement ?

- Observation:

$$Sum(1 \ldots n) = Sum(1 \ldots n\text{-}1) + A[n],$$
$$Sum(2 \ldots n) = Sum(2 \ldots n\text{-}1) + A[n],$$
$$\vdots$$
$$Sum(n\text{-}1 \ldots n) = Sum(n\text{-}1 \ldots n\text{-}1) + A[n],$$

- Maximum($Sum(1 \ldots n), Sum(2 \ldots n), \ldots, Sum(n\text{-}1 \ldots n)$)

  =

  Maximum($Sum(1 \ldots n\text{-}1), Sum(2 \ldots n\text{-}1), \ldots, Sum(n\text{-}1 \ldots n\text{-}1)$) + $A[n]$

- We have converted it to another problem on first *n-1* numbers

Max subarray sum: subproblem

- Improvement ?
- Observation:

$$Sum(1 \ldots n) = Sum(1 \ldots n\text{-}1) + A[n],$$
$$Sum(2 \ldots n) = Sum(2 \ldots n\text{-}1) + A[n],$$
$$\vdots$$
$$Sum(n\text{-}1 \ldots n) = Sum(n\text{-}1 \ldots n\text{-}1) + A[n],$$

Push it to the subproblem

- Maximum($Sum(1 \ldots n), Sum(2 \ldots n), \ldots, Sum(n\text{-}1 \ldots n)$)

=

Maximum($Sum(1 \ldots n\text{-}1), Sum(2 \ldots n\text{-}1), \ldots, Sum(n\text{-}1 \ldots n\text{-}1)$) + $A[n]$

- We have converted it to another problem on first $n\text{-}1$ numbers

Asking subproblem to do more

- Subproblem: max_sum_{n-1} and $max\_suffix\_sum_{n-1}$

- $max\_suffix\_sum_{n-1}$ is defined as
  Maximum($Sum(1 \ldots n\text{-}1), Sum(2 \ldots n\text{-}1), \ldots Sum(n\text{-}1 \ldots n\text{-}1)$)

- $$max\_sum_n = \text{Maximum}(\, max\_sum_{n-1},\; Sum(1 \ldots n\text{-}1) + A[n],\; Sum(2 \ldots n\text{-}1) + A[n],\; \vdots\; Sum(n\text{-}1 \ldots n\text{-}1) + A[n],\; A[n]\,)$$

- $$= \text{Maximum}(\, max\_sum_{n-1},\; max\_suffix\_sum_{n-1} + A[n],\; A[n]\,)$$

Asking subproblem to do more

- Subproblem: max_sum_{n-1} and $max\_suffix\_sum_{n-1}$
- $max\_sum_n = \text{Maximum}(\ max\_sum_{n-1},$
  $max\_suffix\_sum_{n-1} + A[n],$
  $A[n]\ )$
- We are asking the subproblem to compute max_suffix_sum for size $n\text{-}1$
  So, we also need to compute max_suffix_sum for size n
- $max\_suffix\_sum_n = ?$
- $max\_suffix\_sum_n = \text{Maximum}(\ max\_suffix\_sum_{n-1} + A[n],$
  $A[n]\ )$

$T(n) = T(n\text{-}1) + O(1) \quad \Rightarrow T(n) = O(n)$

Maximum subarray sum

O(n) algorithm:

$max\_sum \leftarrow 0$; $max\_suffix\_sum \leftarrow 0$;

for ($i \leftarrow 1$ to n){

$max\_sum \leftarrow$ maximum(max_sum, $max\_suffix\_sum + A[i]$, $A[i]$);

$max\_suffix\_sum \leftarrow$ maximum($max\_suffix\_sum + A[i]$, $A[i]$);

}

Alternate implementation

$max\_sum \leftarrow 0$; $max\_suffix\_sum \leftarrow 0$;

for ($i \leftarrow 1$ to n){

$max\_suffix\_sum \leftarrow \text{maximum}(A[i], max\_suffix\_sum + A[i])$;

$max\_sum \leftarrow \text{maximum}(max\_suffix\_sum, max\_sum)$;

}

- Here we are updating the two variables in a different order.

Reducing to a subproblem

- When solving a problem recursively/inductively, it is sometimes useful to solve a more general problem
- Stronger induction hypothesis

Exercises

- Given share prices for *n* days
  $p_1, p_2, \ldots, p_n$

- You have to buy it on one of the days and sell it on a later day.

- Maximum profit possible in $O(n)$?

- $\max_{\{j > i\}} (p_j - p_i)$

Exercises

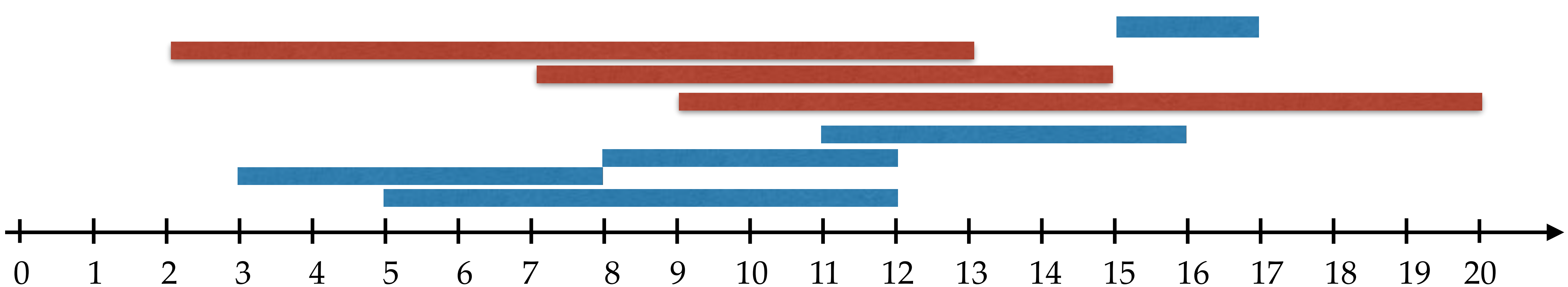
- There is a party with n people, among them there is 1 celebrity.
- A celebrity is someone who is known to everyone, but she does not know anyone.
- you ask the any person i if they know person j.
- Can you do find the celebrity in $O(n)$ queries?

Sign up on Piazza

- https://piazza.com/iit_bombay/spring2024/cs218
- access code: cs218

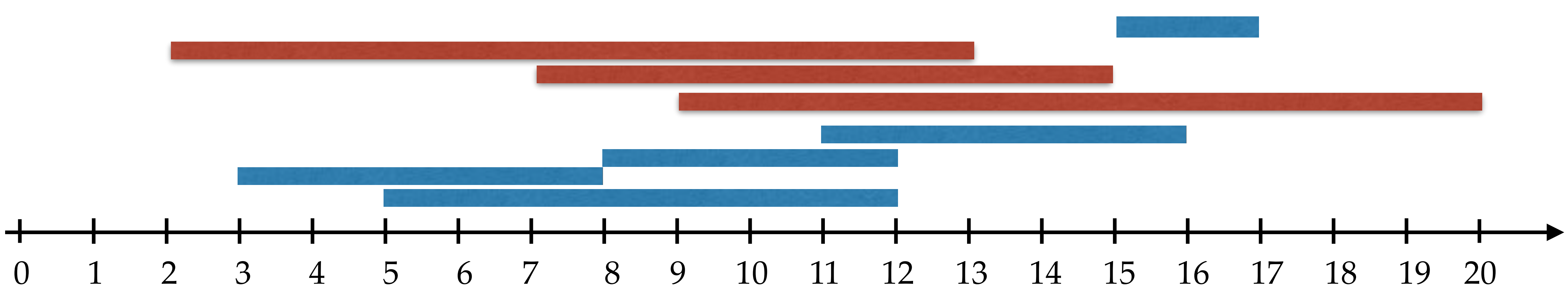
Interval containment

- Given a set of intervals
  count the number of intervals which are not contained in any other interval.
- (5, 12), (3, 8), (8, 12), (11, 16), (9, 20), (15, 17), (7, 15), (2, 13)
- (9, 20), (7, 15), (2, 13)



Interval containment

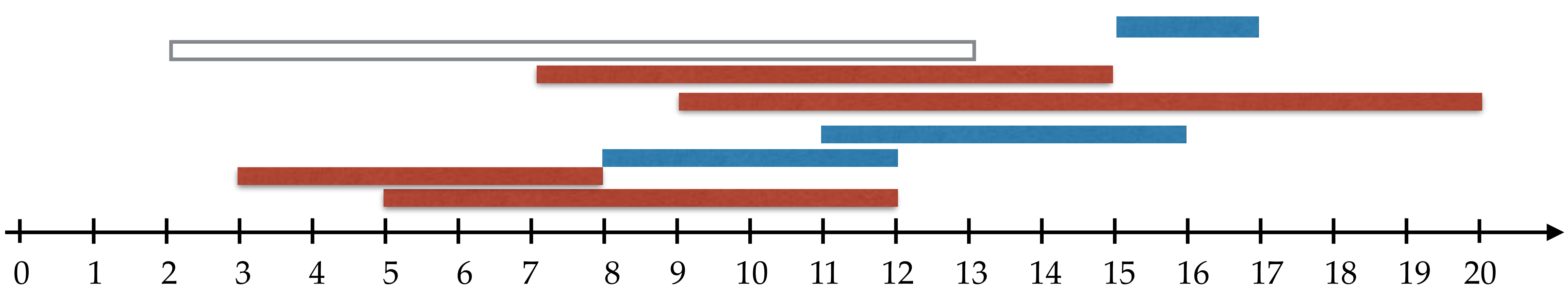
- Given a set of intervals
  count the number of intervals which are not contained in any other interval.
- Naive solution: for every interval, check every other interval
- $O(n^2)$



Subproblem

- Suppose we have a solution for first *n-1* intervals.

- (5, 12), (3, 8), (8, 12), (11, 16), (9, 20), (15, 17), (7, 15), (2, 13)
- (3,8), (5, 12), (9, 20), (7, 15)

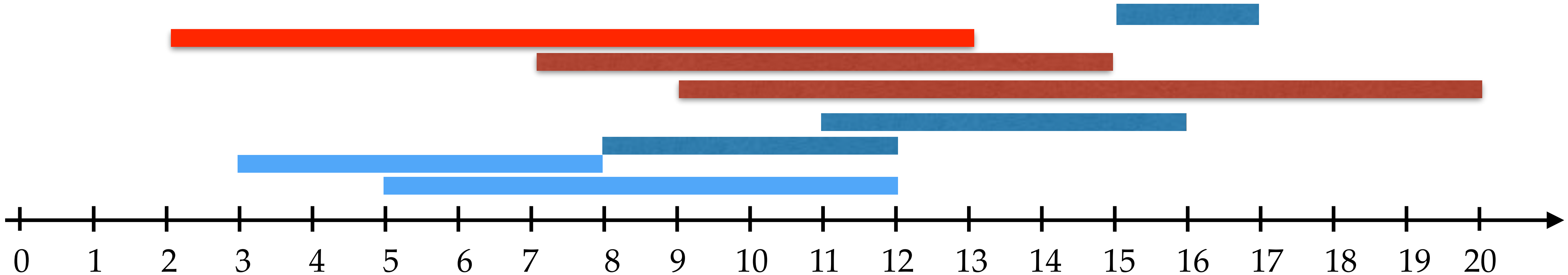


Interval containment

- When we introduce the nth interval
  need to check if it is contained in any other interval
  or if it contains other intervals. Seems to take $O(n)$ time.

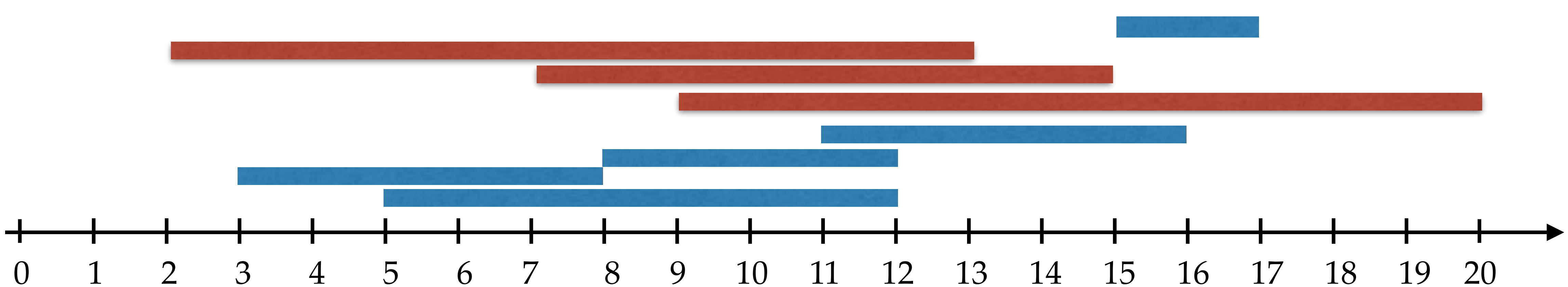
- (5, 12), (3, 8), (8, 12), (11, 16), (9, 20), (15, 17), (7, 15), (2, 13)

- ~~(3,8)~~, ~~(5, 12)~~, (9, 20), (7, 15), (2, 13)



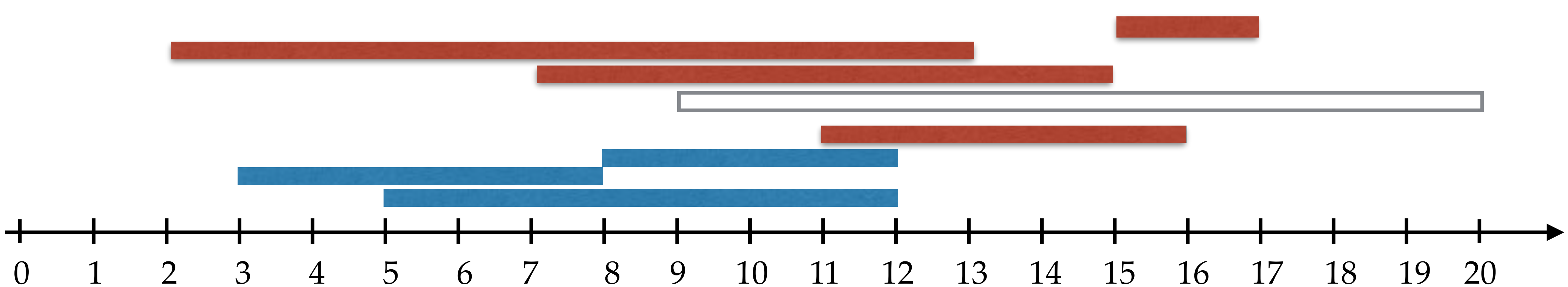
Reordering input

- Can we reorder the intervals so that the work at the last step reduces?
- Two possible orders: increasing start time, increasing finish time
  - (2, 13), (3, 8), (5, 12), (7, 15), (8, 12), (9, 20), (11, 16), (15, 17),
  - (3, 8), (5, 12), (8, 12), (2, 13), (7, 15), (11, 16), (15, 17), (9, 20),



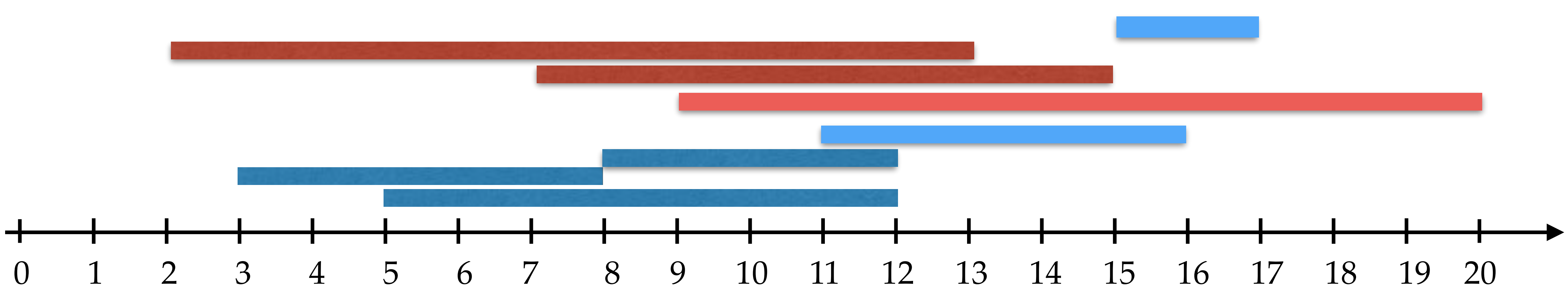
Increasing finish time

- Can we reorder the intervals so that the work at the last step reduces?
- Consider increasing finish time
  - (3, 8), (5, 12), (8, 12), (2, 13), (7, 15), (11, 16), (15, 17), (9, 20),
  - (2, 13), (7, 15), (11, 16), (15, 17),



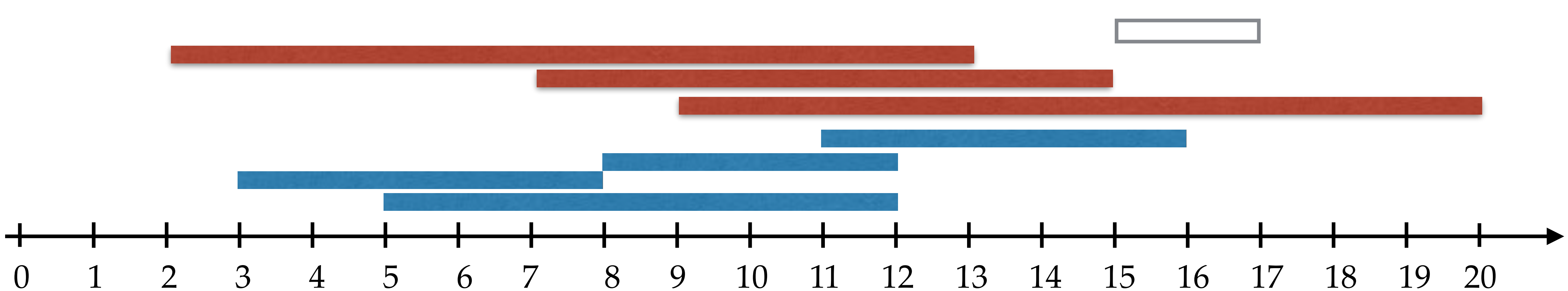
Increasing finish time

- Can we reorder the intervals so that the work at the last step reduces?
- Consider increasing finish time. Same $O(n)$.
  - (3, 8), (5, 12), (8, 12), (2, 13), (7, 15), (11, 16), (15, 17), (9, 20),
  - (2, 13), (7, 15), ~~(11, 16)~~, ~~(15, 17),~~ (9, 20),



Increasing start time

- Consider increasing start time
  - (2, 13), (3, 8), (5, 12), (7, 15), (8, 12), (9, 20), (11, 16), (15, 17),
  - The last interval cannot contain any other
  - Need to check whether the last interval is contained in any other
  - Same as whether any previous interval finishes after the last one.



Algorithm

- Maintain the highest finish time among interval seen so far
- Go over all intervals in increasing order of start times.
- For an interval (s_i, f_i):
  - if $f_i > largest\_finish\_time$ then
    - number_of_maximal_intervals ++ ;
    - $largest\_finish\_time \leftarrow f_i$
- *O(n log n) time*

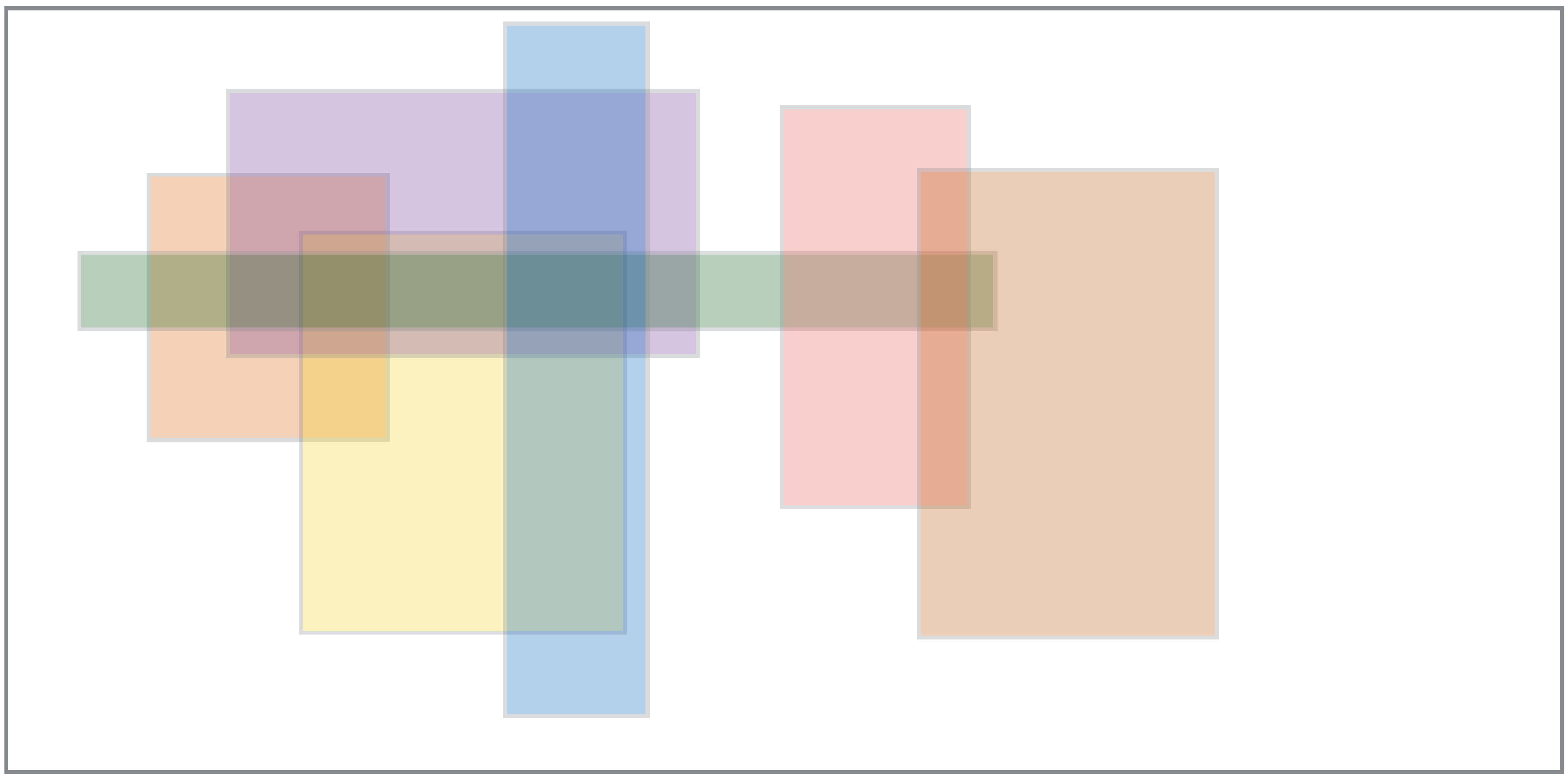
Ideas so far

- Reducing to a subproblem
- Stronger induction hypothesis: Sometimes may need to solve more than what is asked for
- Reordering the input can be helpful.

Divide and Conquer

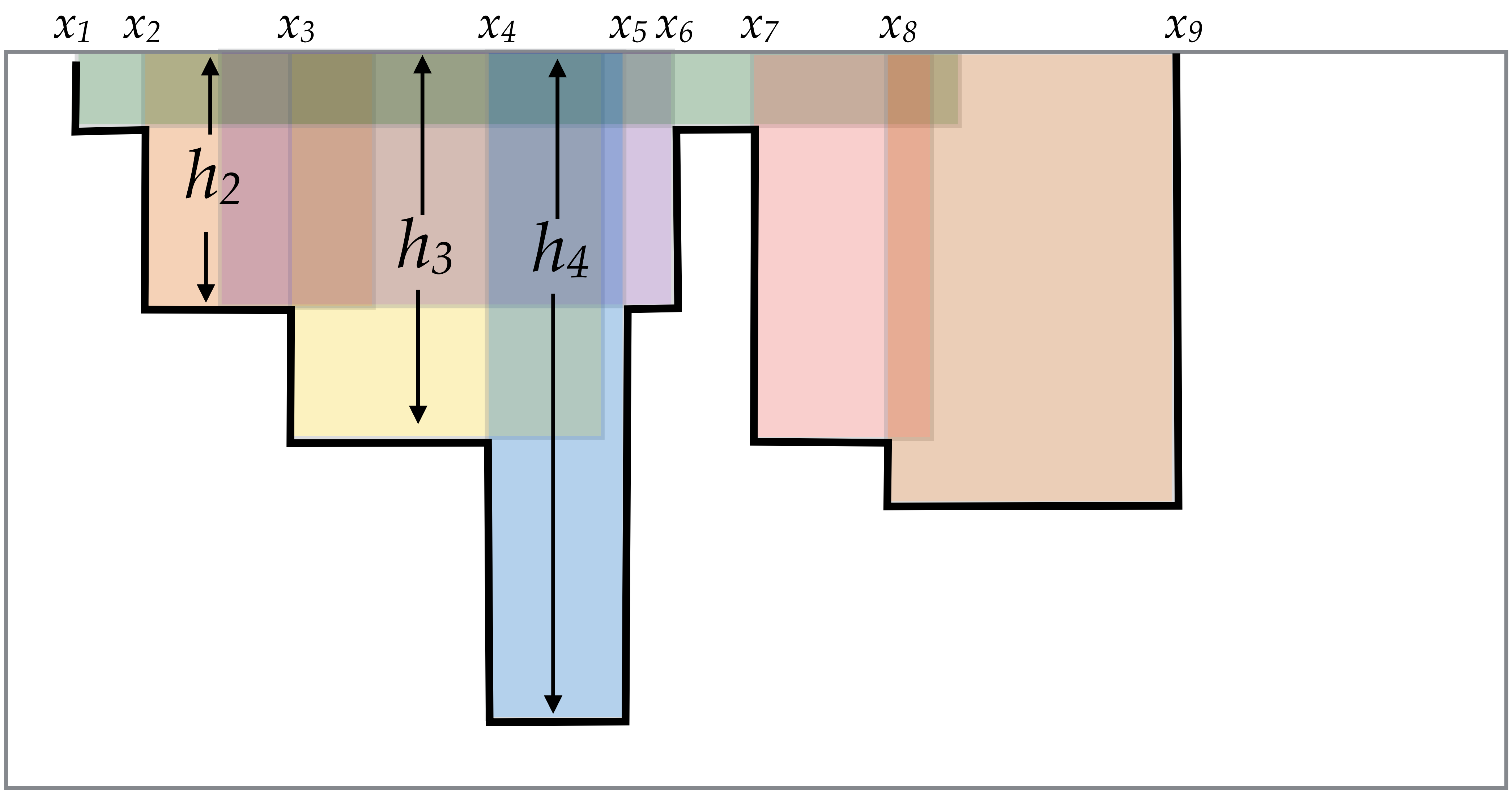
- Classic example: merge sort
- Divide the problem into two (or more) subproblems of size $n/2$
- Combine the solutions of the subproblems and build a solution for the original problem
- $T(n) = a\,T(n/2) + f(n)$
- Divide and conquer might give a running time improvement e.g., from $O(n^2)$ to $O(n \log n)$.

Area coverage



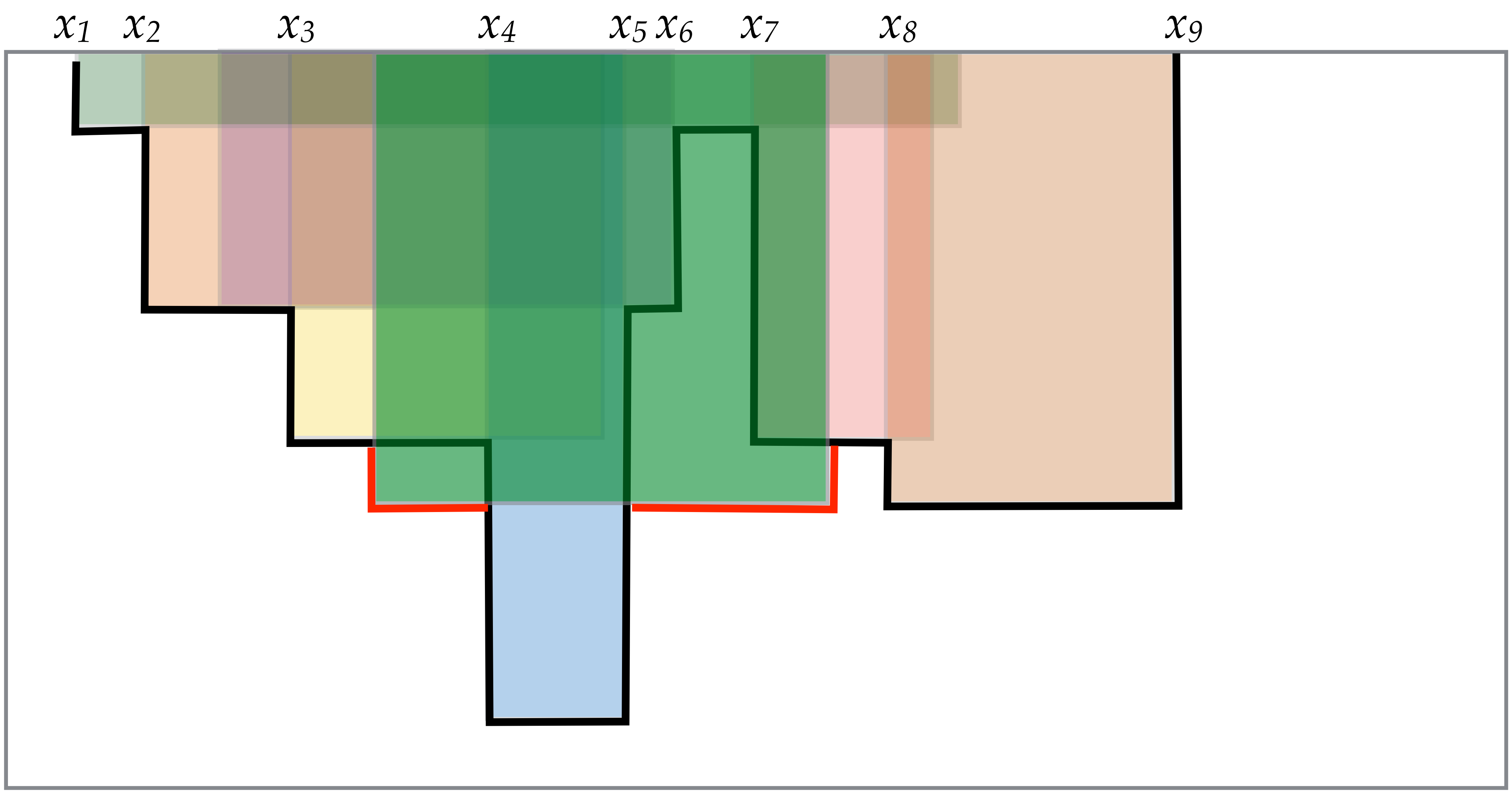
- Find the total area covered

Simpler version



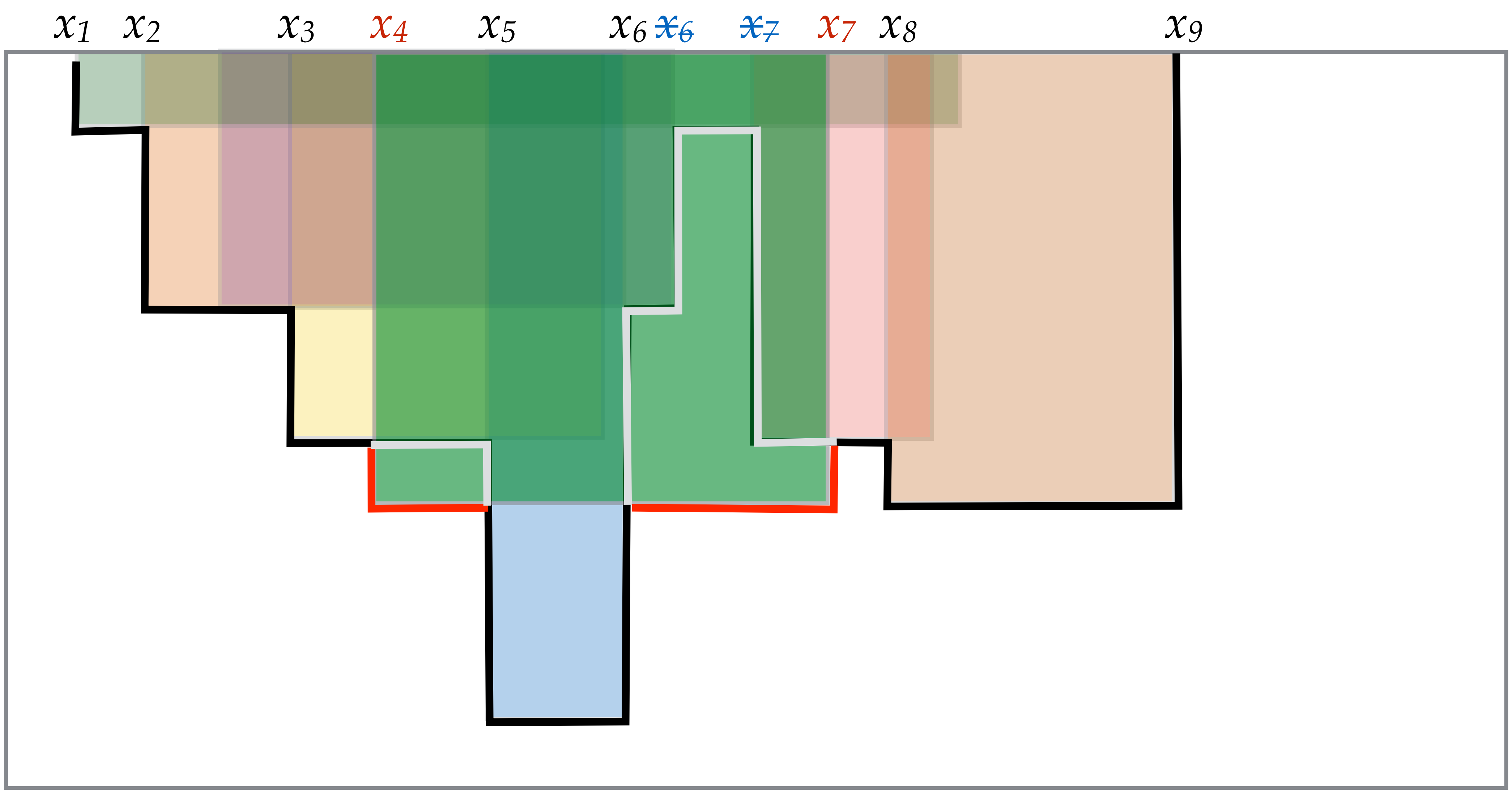
- Also known as skyline problem
- Input: For each rectangle (l_i, r_i, d_i).
- Compute outline: (x_1, h_1), (x_2, h_2), (x_3, h_3), (x_4, h_4), (x_5, h_5), (x_6, h_6), (x_7, h_7), (x_8, h_8), $(x_9, 0)$,

O(n^2) algorithm



- First solution: introduce rectangles one by one, and update the outline.
- Time: *O(n)* per update.

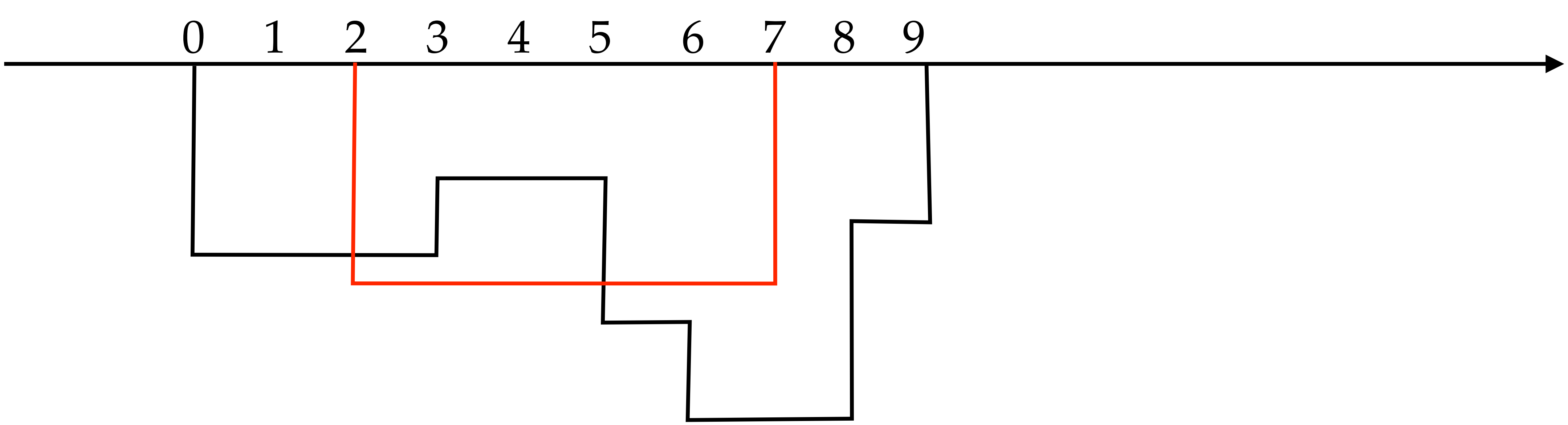
$O(n^2)$ algorithm



- First solution: introduce rectangles one by one, and update the outline.
- Time: $O(n)$ per update.

Update example

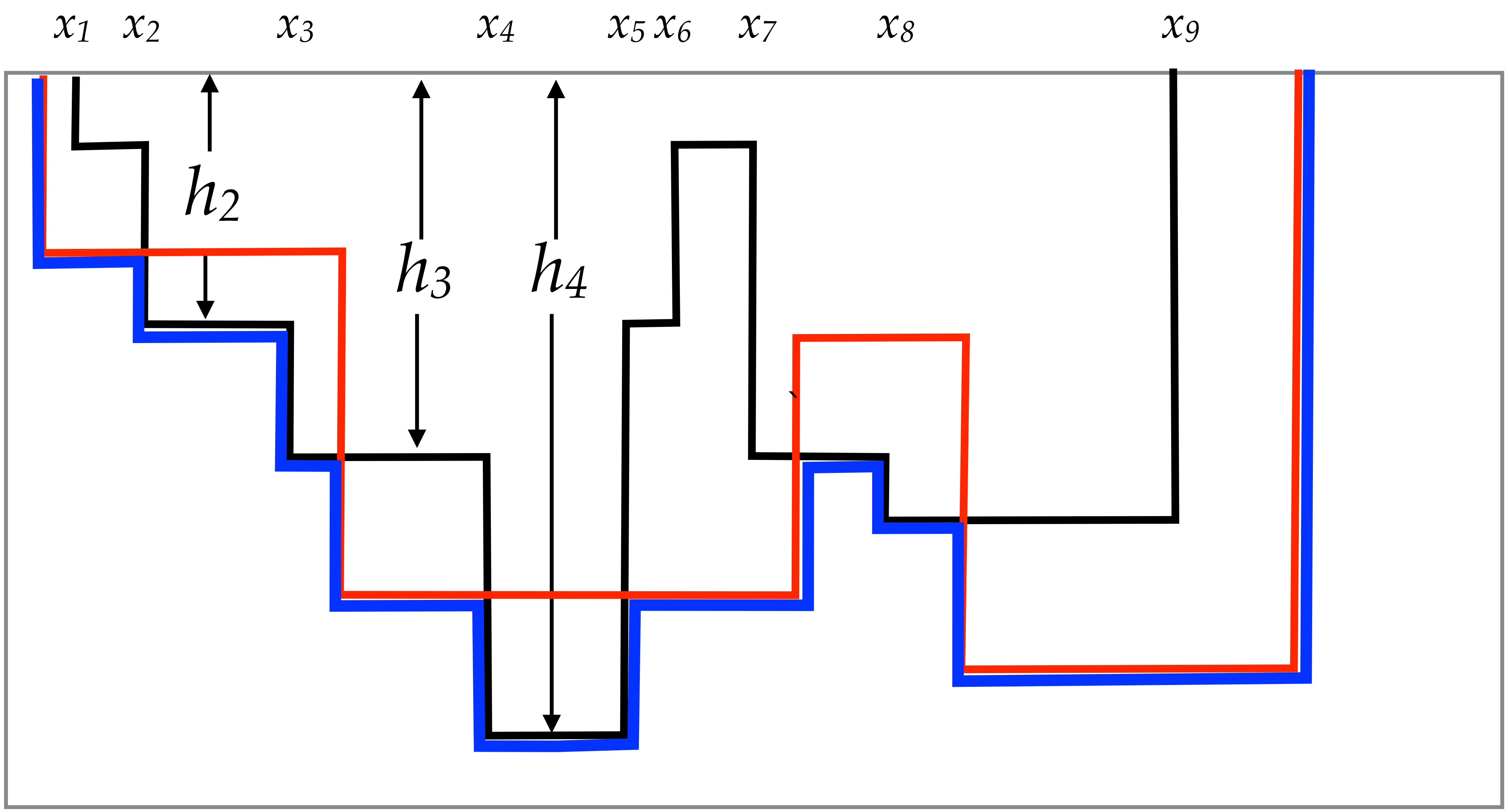
- Current outline: (0, 5), (3, 3), (5, 7), (6, 9), (8, 4), (9, 0)
- Incoming rectangle: (2, 7, 6)
- Updated outline: (0, 5), (2, 6), ~~(3, 3),~~ (5, 7), (6, 9), (8, 4), (9, 0)
- Update can be done in *O(n)* time?



Divide and conquer approach

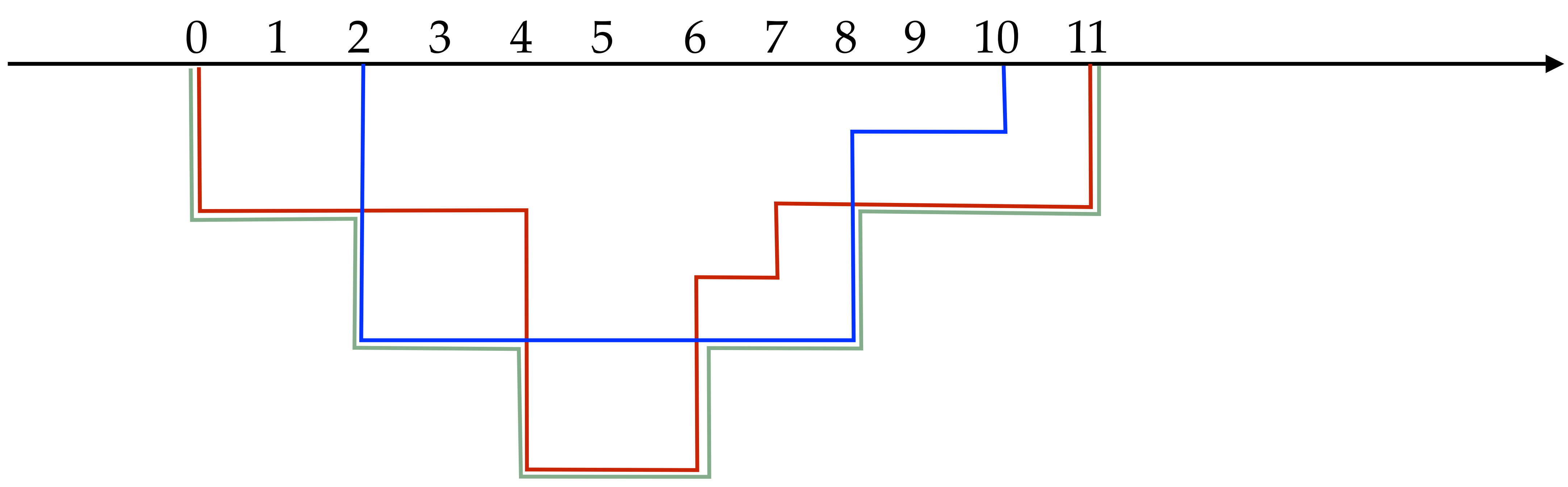
- Divide the set of rectangles into two parts with $n/2$ rectangles each.
- Suppose we have computed the outline for each set of $n/2$ rectangles.
- Outline 1: $(x_{1,} h_1), (x_{2,} h_2), (x_{3,} h_3), \ldots, (x_{m,} 0)$
- Outline 2: $(a_{1,} p_1), (a_{2,} p_2), (a_{3,} p_3), \ldots, (a_{k,} 0)$
- "merge" the two outlines to compute a new outline.

Merge two outlines



Merge two outlines

- Outline 1: (0, 2), (4, 6), (6, 3), (7, 2), (11, 0)
- Outline 2: (2, 4), (8, 1), (10, 0)
- Merged: (0, 2), (2, 4), (4, 6), (6, 4), (8, 2), (11, 0)



Merge two outlines

- Outline 1: $(x_{1,}\, y_1)$, $(x_{2,}\, y_2)$, $(x_{3,}\, y_3)$, …, $(x_{m,}\, 0)$
- Outline 2: $(a_{1,}\, b_1)$, $(a_{2,}\, b_2)$, $(a_{3,}\, b_3)$, …, $(a_{k,}\, 0)$
- Pointers for two queues i and j.
- Maintain the current height in each outline
  height_1, height_2
- If $x_i < a_j$ then
  - *height_1* $\leftarrow y_i$
  - Push $(x_i\ , max\ (\ height\_1, height\_2\ )\ )$
  - $i \leftarrow i + 1$
- Else is similar
- Final clean up: remove consecutively repeating heights in the merged outline

Alternative implementation

- Maintain the current height in each outline *height_1, height_2*
- If $x_i < a_j$ then
  - *height_1* $\leftarrow y_i$
  - *height_to_push* $\leftarrow$ *max (height_1, height_2)*
  - If *(last_pushed_height* $\neq$ *height_to_push)*
    - Push $(x_i$, *height_to_push)*
    - *last_pushed_height* $\leftarrow$ *height_to_push*
  - $i \leftarrow i + 1$
- Else is similar

Other approaches

- Divide and conquer with respect to heights?
- Approaches without divide and conquer
- Reordering the input
    - Increasing order of left end points: $O(n \log n)$ time implementation possible using a data structure like balanced binary tree.
    - Decreasing order of heights: $O(n \log n)$ time implementation possible using a data structure like balanced binary tree or heap.
- $O(n \log n)$ necessary ?

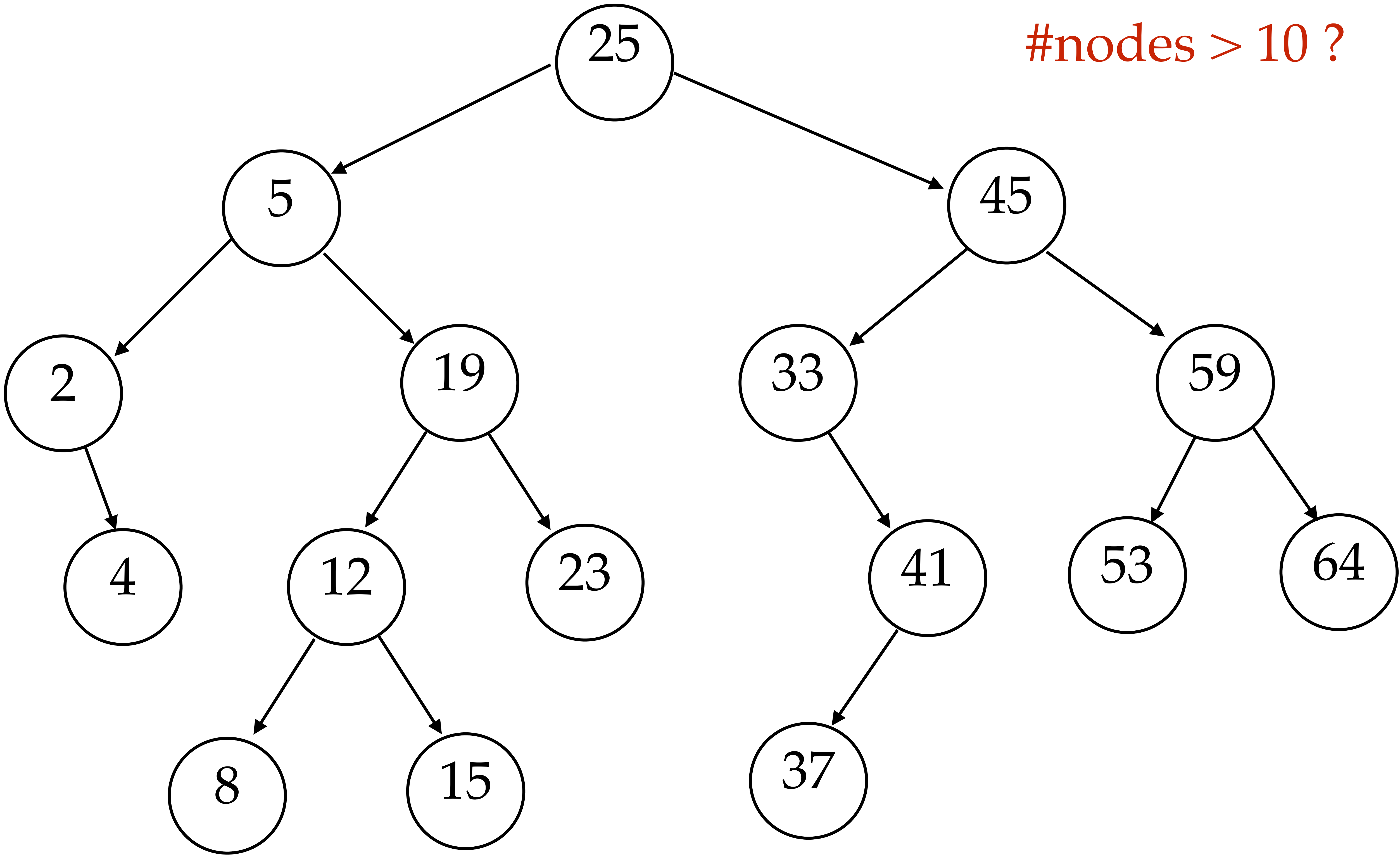
Significant inversion

- Given an array A of integers
- a pair (i, j) is called a significant inversion if $i < j$ and $A[i] > 2A[j]$.
- $O(n \log n)$ time algorithm to find the number of significant inversions.
- Divide and conquer will work.
- Alternate approach.

Significant inversion

- a pair (i, j) is called a significant inversion if $i < j$ and $A[i] > 2A[j]$.
- For any j, count how many are $> 2A[j]$ among first j-1 numbers.
- Easy to count if first j-1 numbers are in a sorted array
- But to maintain sorted array we will need $O(n)$ per insertion.
- What other data structure can be used?

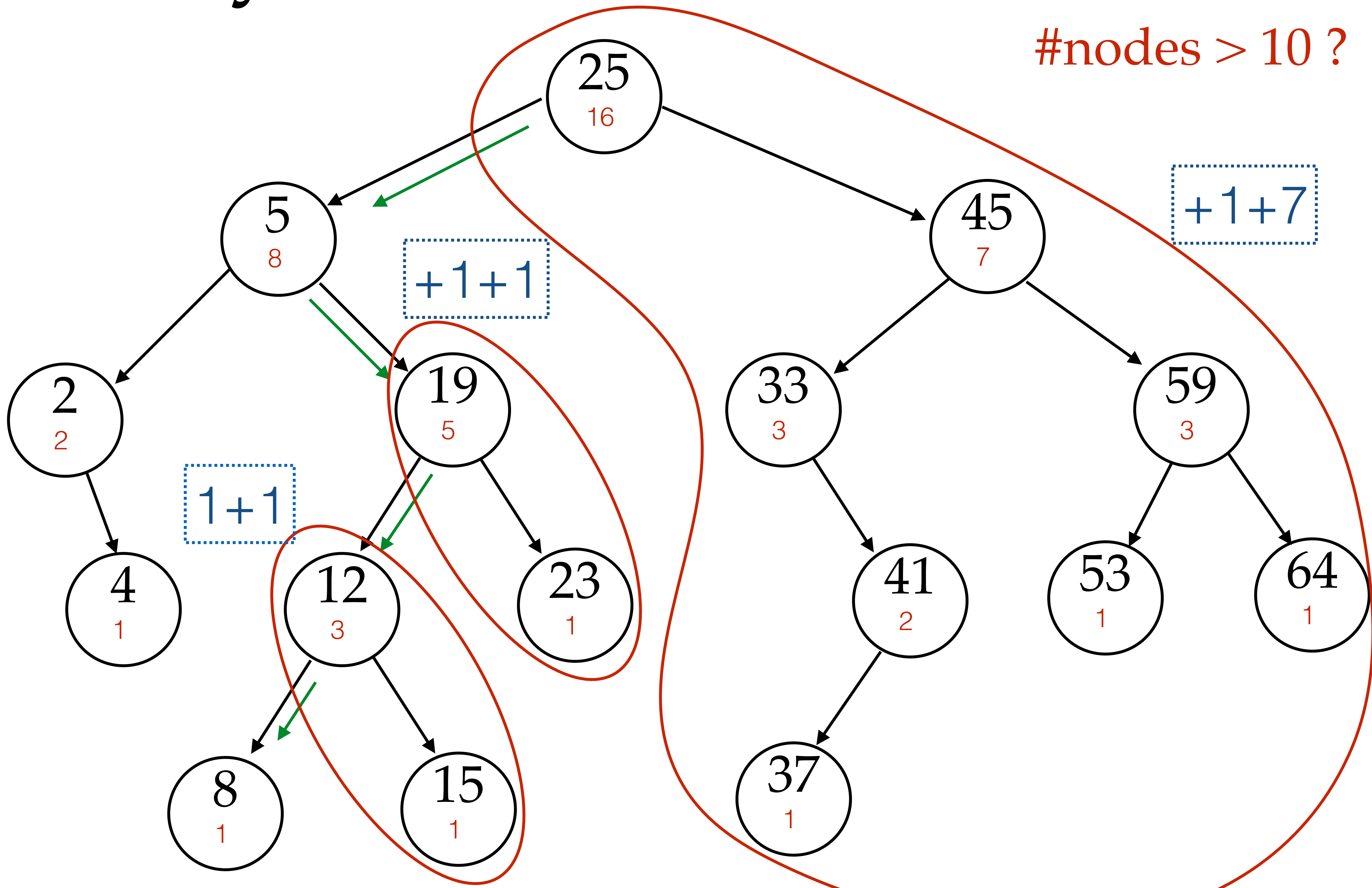
Binary search tree



Counting in BST

- Counting number of $nodes > x$ seems to take $O(n)$ time in the standard BST.
- What if store for each node, the number of nodes greater than itself ?
- Too much time to update these numbers on insertion of a new element.
- Correct idea: Store for each node x, the size of the subtree rooted at x

Binary search tree with count



Counting in BST

- Store for each node x, the size of the subtree rooted at x
- Can we maintain this information on insertion of a new element?
- Need to update the counts for all nodes on the path from the root to the new node.
- $O(\log n)$ for balanced binary tree.
- Can counts be maintained in the re-balance step?
- Can you do range count: *# nodes* in range l to r
- Other data structures?

Algorithm: Significant inversion

- Maintain a balanced BST. Initially empty.
  For each node x, we need to store size of the subtree rooted at x.
- *no_of_pairs* $\leftarrow$ *0;*
- For $j \leftarrow 1$ to n
  - Look for the right position p for $2A[j]$ in the BST (no insertion).
  - For every node x on the path from p to the root:
    - If ($x > 2A[j]$) *then*
      Add (1+ size of the subtree rooted at right child of x) to *no_of_pairs*
  - Insert $A[j]$ in the BST and increase the size count for every node on the path from root to $A[j]$ by 1

Exponentiation

- Given a and n, compute a^n.
- $a \times a \times \cdots \times a$ $\quad$ n times
- n-1 multiplications
- $a^{16} = (((a^2)^2)^2)^2$ only 4 multiplications
- $a^{11} = ((a^2)^2)^2 \times a^2 \times a$ only 5 multiplications
- Repeated squaring technique

Repeated Squaring

- Exp(a, n) :
  - If n is even
    - return (Exp(a, $n/2$) $)^2$
  - If n is odd
    - return (Exp(a, $(n\text{-}1)/2$) $)^2 \times a$
  - If n is 1
    - return a

Number of multiplications

$$T(n) \le T(n/2) + 2$$

$$T(n) \le 2 \log n$$

Another implementation

- Exp(a, n) :
  - If n is even
    - return (Exp(a^2, $n/2$))
  - If n is odd
    - return (Exp(a^2, $(n-1)/2$)) $\times$ a
  - If n is 1
    - return a

Iterative implementation

- Input: a, n
- Initialize
  $a\_power\_n \leftarrow 1;$ // this will be a^n at the end
  $a\_two\_power \leftarrow a;$ // this will be $a^{2^\wedge i}$ after i iterations
- while $(n > 0)$
  - If (n is odd) then $a\_power\_n \leftarrow a\_two\_power \times a\_power\_n;$
  - $a\_two\_power \leftarrow a\_two\_power \times a\_two\_power;$
  - $n \leftarrow n/2$ //integer part after division by 2 //or right-shift by 1 bit

Repeated squaring

- Does it give the minimum number of multiplications?
- What about a^{15} ?
  - can be done in 5 multiplications
- Given n, what the minimum number of multiplications required for a^n ? No easy answer.
- Can apply repeated squaring for other operations like Matrix powering.
- Apparently proposed by Pingala (200 BC ?).

Fibonacci numbers

- $F(n) = F(n\text{-}1) + F(n\text{-}2)$
- Used by Pingala to count the number of patterns of short and long vowels.
- Here again we are repeating the same operation n times.
- Can repeated squaring be used?

Integer Multiplication