CS218 Design and analysis of algorithms **Jan-Apr 2023**

Exercises: Greedy, Dynamic Programming (No submission)

Lecture 7 & 8: Greedy algorithms

1. Let $G(V, E)$ be a graph with edge weights and $V = V_1 \cup V_2$ be a partition of vertices. Let C be the set of cut edges connecting V_1 with V_2, i.e.,

   $$C = \{(u, v) : u \in V_1, v \in V_2\}.$$

   Let e^* be the minimum weight edge in C. Prove that there must exists a minimum weight spanning tree containing e^*.

2. Let $G(V, E)$ be a graph with edge weights and let v_1 be an arbitrary vertex. Let e^* be the minimum weight edge incident on v_1. Prove that there must exists a minimum weight spanning tree containing e^*.

3. Suppose you are given a set of n course assignments today, each of which has its own deadline. Let the i-th assignment have deadline d_i and suppose to finish the i-th assignment it takes ℓ_i time. Given that there are so many assignments, it might not be possible to finish all of them on time. If you finish an assignment at time t_i which is more than its deadline d_i, then the difference $t_i - d_i$ is called the lateness of this assignment (if $t_i < d_i$ then the lateness is zero). Since you want to maintain a balance among courses, you want that the maximum lateness over all assignments is as small as possible. You want to find a schedule for doing the assignments which minimizes the maximum lateness over all assignments. Can you show that a greedy algorithm will give you an optimal solution?

   - Greedy Strategy 1: Do the assignments in increasing order of their lengths (ℓ_i).
   - Greedy Strategy 2: Do that assignment first whose deadline is the closest.
   - Greedy Strategy 3: Do that assignment first for which $d_i - \ell_i$ is the smallest.

   Two of these strategies don't work. Give examples to show that they don't work. One of the strategies actually work. Prove that it works by arguing that there is an optimal solution which agrees with the first step of the greedy algorithm.

   Example: $d_1 = 20, \ell_1 = 10$, $d_2 = 40, \ell_2 = 20$, $d_3 = 60, \ell_3 = 30$. If the assignments are done in order $(1, 3, 2)$ then the maximum lateness will be 20 (for assignment 2). If the assignments are done in order $(1, 2, 3)$ then the maximum lateness will be 0.

4. Consider another variant. Now, all assignments are equally long, so let's say each takes a unit time to finish. The ith assignment has deadline d_i and a reward r_i. You get the reward only if you finish the assignment within the deadline, otherwise the reward is zero. Design an algorithm to find the maximum possible reward you can get.

5. Hard problem. Consider another variant. For each assignment, you know its deadline d_i and the time ℓ_i it takes to finish it. Suppose you get zero marks for finishing an assignment after its deadline. So, either you should do the assignment within the deadline or not do it at all. How will you find the maximum number of assignments possible within their deadlines.

6. Given a set of intervals, you need to assign a color to each interval such that no two intersecting intervals have the same color. Design an efficient algorithm find a coloring with minimum number of colors. To put the problem in another way, given arrival and departure times of trains at a station during the day, what is the minimum number of platforms that is sufficient for all trains.

7. Given a list of n natural numbers $d_1, d_2, \ldots, d_n$, we want to check whether there exists an undirected graph G on n vertices whose vertex degrees are precisely $d_1, d_2, \ldots, d_n$ (that is the ith vertex has degere d_i) and construct such a graph if one exists. G should not have multiple edges between the same pair of vertices and should not have self-loop (edges having same vertex as the two endpoints).

   Example 1: (2, 1, 3, 2) . The graph with the set of edges $(v_1, v_3), (v_1, v_4), (v_2, v_3), (v_3, v_4)$ has this degree sequence.

   Example 2: (3, 3, 1, 1) . There is no graph on four vertices having these degrees.

8. Suppose you are an advertisement company who wants to advertise something to all n people in the city. You know that each of these n people will come to the city center on Sunday for some interval of time. You have acquired these time intervals for all people through some unethical means. You cannot put ads at the city center, but you can pay people to carry your ad on them (maybe by wearing a t-shirt). Assume that if X is carrying the ad, then anyone whose time interval intersects with the time interval of X will see the ad (of course, X will also see the ad). You want to choose minimum number of people to whom you should pay so that everyone sees the ad. Design an algorithm for this and prove its correctness.

Lecture 9: Dynamic Programming

Flavour 1: where the subproblems are on suffixes/prefixes of the input.

1. Recall the discussion on the problem of finding a set of disjoint intervals that maximizes the total length.

   - Suppose we sort the intervals in increasing order of their finishing times. And then consider two kinds of solutions: solutions containing the first interval and solutions not containing the first interval. Construct an example where the number of distinct recursive calls become exponentially large.

     Alternatively, suppose we work with the same order. But, now we consider two kinds of solutions. Solutions containing the last interval and solutions not containing the last interval. Do you think now there will be only polynomially many recursive calls?

   - The algorithm we described in the class only computed the optimal cost. Update the pseudocode to also output an optimal set of intervals.

2. You are going on a car trip from city A to city B that will take multiple days. On the way, you will encounter many cities. You plan to drive only during the day time and on each night you will stay in one of the intermediate cities. Suppose you can drive at most d kilometers in a day. You are given the distances of the intermediate cities from city A, say, $d_1, d_2, \ldots, d_n$. And distance of B from A is d_{n+1}. You are also given the costs of staying in various cities for one night, say, $c_1, c_2, \ldots, c_n$. Find a travel schedule, that is, in which all cities you should do a night stay, such that your total cost of staying is minimized.

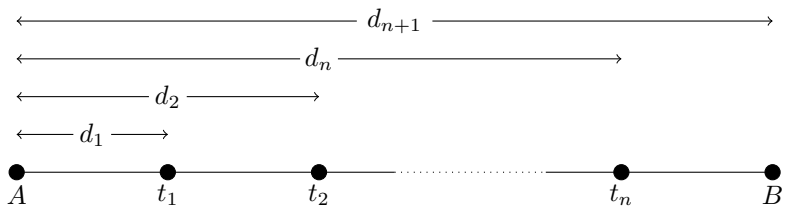


Figure 1

3. Given an array of integers, you want to find a subset with maximum total sum such that no two elements in the subset are adjacent. For example, for the array $\{6, 4, 3, 2, 1, 5\}$, the desired subset is

$\{6, 3, 5\}$ with total sum 14. Design an $O(n)$-time algorithm for this problem, where n is the length of the array.

4. We are given a directed graph, where each edge goes from a lower index vertex to a higher index vertex. Want to find the longest path from vertex 1 to vertex n.

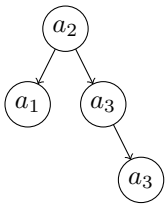
5. Given a sequence of numbers, we want to find the longest increasing subsequence.

Flavour 2: where the subproblem is on a substring of the input.

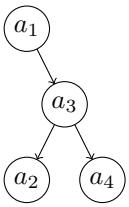
1. The naive algorithm to multiply two matrices of dimensions $p \times q$ and $q \times r$ takes time $O(pqr)$. Suppose we have four matrices A, B, C, D which are $2 \times 4, 4 \times 3, 3 \times 2, 2 \times 5$ respectively. If you multiply $ABCD$ in the order $(AB)(CD)$, it will take time $2 \times 4 \times 3 + 3 \times 2 \times 5 + 2 \times 3 \times 5 = 84$, on the other hand if you multiply in the order $A((BC)D)$, it will take time $4 \times 3 \times 2 + 4 \times 2 \times 5 + 2 \times 4 \times 5 = 104$.

   Given matrices $A_1, A_2, \ldots, A_n$ with dimensions $p_1 \times p_2, p_2 \times p_3, \ldots, p_n \times p_{n+1}$, design an algorithm to find the order in which you should multiply $A_1 A_2 \cdots A_n$ which minimizes the multiplication time.

2. Suppose you have n keys $a_1 < a_2 < \cdots < a_n$ and you want to build a binary search tree that allows efficient search for these keys. If the search queries are distributed uniformly across the keys then it would make sense to build a balanced binary tree. However, if the some keys are more frequent than others then a balanced binary tree might not be the most efficient structure. Consider an example with four keys $a_1 < a_2 < a_3 < a_4$ with the binary search tree shown in Figure 2a.



(a) A binary search tree



(b) A binary search tree

   Suppose their search frequencies are distributed like $0.7, 0.1, 0.1, 0.1$, respectively. Naturally, we can assume that the search time for a key is proportional to its depth (distance from the root). Then, the average search time for this search tree (Figure 2a) will be $0.7 \times 2 + 0.1 \times 1 + 0.1 \times 2 + 0.1 \times 3 = 2$.

   Now, consider another search tree in Figure 2b. The average search time for this tree will be $0.7 \times 1 + 0.1 \times 2 + 0.1 \times 3 + 0.1 \times 3 = 1.5$, which is better than the first search tree.

   Given search frequencies of n keys, say $f_1, f_2, \ldots, f_n$, we want to find the binary search tree that minimizes the average search time defined as $\sum_i f_i d_i$, where d_i is the depth of the node containing a_i.

Flavour 3: where the subproblem (recursive call) has two different parameters.

1. A subsequence of a string is obtained by possibly deleting some of the characters without changing the order of the remaining ones. For example, *ephn* is a subsequence of *elephant*.

   You are given a string A of length n and another string B of length m ($\leq n$). If we want to check whether A contains B as a subsequence, there is greedy algorithm for it: For $1 \leq i \leq m$, match the character $B[i]$ with its first occurrence in A after the matching of $B[i-1]$.

   For example, if $A = bacbcbabcacba$ and $B = bcbca$, then the greedy approach will match B as follows (shown in red).

   bacbcbabcacba

   Now, suppose to match with the j-th character in string A, you have to pay a cost p_j. And to match B with a subsequence in A, you have to pay the sum of costs of the matched characters.

In the above example, if the prices for the characters in A were $2, 5, 3, 1, 2, 5, 3, 1, 3, 1, 2, 4, 1$ then the matching $bacbcbabcacbaa$ has cost $2 + 3 + 1 + 2 + 3 = 11$. While the matching $bacbcbabcacba$ has cost only $1 + 2 + 1 + 2 + 1 = 7$.

Design an algorithm that given A, B and $p_1, p_2, \ldots, p_n$, can find the minimum cost subsequence in A that can be matched with B. Ideally your algorithm should run in time $O(mn)$.

Lecture 10

1. **Knapsack problem.** Suppose there are n objects with their weights being $w_1, w_2, \ldots, w_n$ and their values being $v_1, v_2, \ldots, v_n$. You need to select a subset of the objects such that the total weight is bounded by W, while the total value is maximized. Your algorithm should run in time $\text{poly}(n, W)$.

   Consider the case when the weights are too large, that is, they are exponential in n. Then the above algorithm is not really efficient. Suppose on the other hand, then values $\{v_1, v_2, \ldots, v_n\}$ are small. Can you design an algorithm running in time $\text{poly}(n, \sum_i v_i, \log W)$?

2. **Solution for problem 4 in Lecture 7,8.** Recall the problem. There are n assignments each of unit length. ith assignment has a deadline d_i and a reward r_i. You get the reward if you do the assignment within its deadline. We want to maximize the total reward.

   Strategy: go over the assignments in decreasing order of rewards. For any assignment, schedule it on the last day available before its deadline. If there is no day available before its deadline then discard it.

   Proof: We start with some optimal solution and try to convert it into our solution step by step, while maintaining optimality.

   Let's say the the optimal solution and our solution agree on the schedule of top $k-1$ assignments (in terms of rewards), for some $k \geq 1$. Consider the kth assignment.

   First consider the case when our solution does not schedule it on any day. This was only because there was no day available after scheduling the top $k-1$. Since the optimal solution agrees with us on the top $k-1$, the optimal solution also cannot schedule the kth assignment on any day.

   Now, consider the case when our solution schedules the kth assignment on day t_k. If the optimal solution schedules it on day s_k then, it must be that $s_k \leq t_k$. This is because after scheduling top $k-1$, the last day available is t_k. Then in the optimal solution, we swap the two assignments scheduled on day s_k and t_k. The kth assignment is still before its deadline. And other assignment gets shifted to an earlier day, which is completely fine. The total reward remains unchanged.

   Another possibility is that the optimal schedule does not schedule the kth assignment on any day. In that case, we throw out the assignment scheduled on day t_k and schedule kth assignment on that day. What we throw out has a lower (or equal) reward than kth assignment. Hence the total reward can only increase.

   Now, the optimal solution agrees with our solution the schedule of kth assignment as well, while maintaining optimality. We repeat the argument for every assignment in our solution (in decreasing order of rewards).

Lecture 11

Flavour 4: where subproblems are parameterized by a prefix and something more

1. Suppose you have a movable shop that you can take from one place to another. You usually take your shop to one of the two cities, say A and B, depending on whichever place has more demand. Suppose you have quite accurate projections for the earnings per day in both the cities for the next n days. However, you cannot simply go to the higher earning city each day because it takes one whole day and costs c to move from one city to the other. For example, if you are in city A on day 5 and want to move to city B, then on day 6 you will have no earnings, you will pay a cost of c and on day 7 you will

have earnings of city B. Design a polynomial time algorithm that takes as input the moving cost ,the earnings per day in the two cities say, $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, and outputs a schedule for the n days that maximizes the total earnings. Assume that you can start with any of the two cities on day 1, without costing anything.

Miscellaneous:

1. Kleinberg Tardos Section 6.3 Segmented least square
2. Kleinberg Tardos Section 6.5 RNA secondary structure
3.
4. Kleinberg Tardos Exercises 10, 16, 28