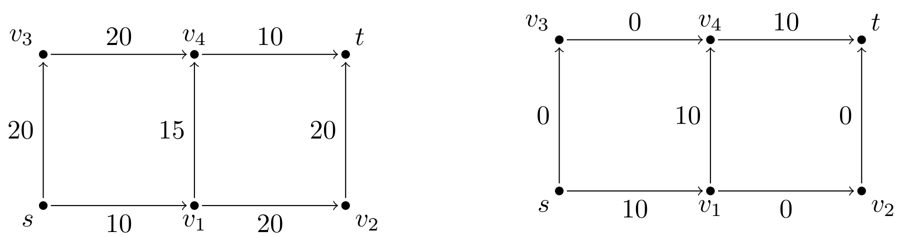
CS218 Design and analysis of algorithms	**Jan-Apr 2023**
Quiz 2	
Total Marks: 20	*Time: 85 minutes*

Instructions.

- Please write your answers concisely.

Que 1. Consider the flow network in Figure (1a) with a source s and a sink t. The edge capacities are shown on the edges. Figure (1b) shows an s-t flow in the same network with flow values indicated on the edges.



(a) A flow network with capacities on the edges

(b) A flow network, with current flow values shown on the edges.

Figure 1: Network Flow

(a) [1 mark]. Suppose the initial flow is given by Figure (1b). Construct the residual graph with respect to this initial flow. Recall that any forward/backward edges with zero capacity are not be kept in the residual graph.

(b) [1 mark]. Is there a path from s to t in the residual graph? Describe the path. What is the bottleneck b (minimum capacity of an edge) on this path?

(c) [1 mark]. Suppose we push b units of flow of along this path. Show the flow network with new flow values.

(d) [1 mark]. Construct the residual graph with respect to the new flow. Is there a path from s to t in this residual graph?

(e) [2 mark]. What is the current outgoing flow from s? Construct an s-t cut in the network whose outgoing capacity is equal to the current outgoing flow from s.

Que 2. Consider the problem of TA (teaching assistants) allocation to courses. A TA may be suitable for a subset of courses and not suitable for other courses. Suppose we are given the suitability information by a $m \times n$ matrix with $\{0, 1\}$ entries, where m is the total number of TAs and n is the number of courses. We are also given the required number of TAs for each course, say, $r_1, r_2, \ldots, r_n$. Naturally, a TA can only be assigned to a course for which they are suitable. And a TA can be assigned to at most one course. We want to find out whether it's possible to meet the TA requirement for every course.

For example, suppose for course X the suitable TAs are $\{1, 2, 3\}$. And for course Y, the suitable TAs are $\{2, 3\}$. Suppose X and Y both require two TAs each. Then it is not possible to meet the requirement for every course.

We plan to solve this problem using the network flow algorithm as a subroutine. For any given input for TA allocation (suitability matrix and r_i's), we want to first build an appropriate flow network. We should design the network in a way that the value of the maximum flow should tell us whether it's possible to meet the TA requirement for every course.

Building the flow network: We create one vertex for every TA and one vertex for every course. We add a (directed) edge from a course to a TA if and only if that TA is suitable for that course. Additionally, we also create a source vertex s and a sink vertex t.

(a) [1 mark]. Where will you put the outgoing edges from s and what will be their capacities?

(b) [1 mark]. Where will you put the incoming edges to t and what will be their capacities?

(c) [1 mark]. What should be the capacity of the edges which are going from a course to a TA?

(d) [1 mark]. If it is possible to meet the TA requirement for every course, then what will be the maximum flow in your network?

(e) [1 mark]. If it is **not** possible to meet the TA requirement for every course, then what can you say about the maximum flow in your network?

(f) [4 marks]. Prove that when it is not possible to meet the TA requirement for every course, then there must be a subset C of courses such that the total number of TAs who are suitable for some course in C is less than the required number $\sum_{i \in C} r_i$. You can directly use the max flow min cut theorem (but not other theorems like Hall's theorem, Dilworth's theorem).

Hint: Say U is a minimum s-t cut (U contains s). Consider the set of courses in U.

Que 3. Damerau-Levenshtein distance between two strings (over any alphabet) is defined as the minimum number of valid operations required to convert one string into another, where the valid operations are – deletion of a character, insertion of a character, substituting one character with another, and swapping of two adjacent characters. There is no restriction on the order in which these operations are performed. For example, the two strings `from` and `north` have distance 4, as shown below.

`from` ↔ `form` ↔ `norm` ↔ `norh` ↔ `north`

We want to design a dynamic programming algorithm to compute the Damerau-Levenshtein distance between two given strings. Let the two given strings be $a_1 a_2 \cdots a_n$ and $b_1 b_2 \cdots b_m$, where each a_i and b_j belong to an alphabet Σ.

Let $d(i,j)$ denote the distance between the substrings $a_1 a_2 \cdots a_i$ and $b_1 b_2 \cdots b_j$, for $0 \le i \le n$ and $0 \le j \le m$ ($i = 0$ or $j = 0$ just means empty substring).

We will build an $(n+1) \times (m+1)$ table D whose (i,j) entry is supposed to be $d(i,j)$ for $0 \le i \le n$ and $0 \le j \le m$. Finally, $D(n,m)$ will be our answer.

Set $D(0,i) = i$ and $D(j,0) = j$ for each i and j.

To fill the rest of the table we use the following equation

$$D(i,j) = \min \begin{cases} D(i-1,j)+1 & \text{(deletion)} \\ D(i,j-1)+1 & \text{(insertion)} \\ D(i-1,j-1) & \text{considered only if } a_i = b_j \\ D(i-1,j-1)+1 & \text{(substitution)} \\ D(i-2,j-2)+1 & \text{considered only if } i,j \ge 2, \text{ and } a_i = b_{j-1} \text{ and } a_{i-1} = b_j \text{ (swapping).} \end{cases}$$

It turns out that this approach is wrong. ☺

Find an example of two strings, where this algorithm will output a larger number than the Damerau-Levenshtein distance (2 marks). Fill up the table D completely for your example, as per the above equation (2 marks). Show the correct distance between the two strings in your example, via a sequence of valid operations (1 mark).

Hint 1: A similar approach would have worked correctly, if we didn't have the swapping operation in our distance definition.

Hint 2: There is an example where both the strings have length at most 3.