Reduction from SAT to IND-SET

SAT \leq IND-SET

This will prove that IND-SET is NP-hard.

**IND-SET**: Given a graph \( G \) and number \( k \), does \( G \) have an independent set of size \( k \)?

**Reduction:** Boolean formula \( \phi \rightarrow \text{Graph } G(\phi), \text{ number } k(\phi) \)

Such that \( \phi \) is satisfiable

if and only if

\( G(\phi) \) has an independent set of size \( k(\phi) \)

Without loss of generality, we will assume that

\( \phi \) is in conjunctive normal form (CNF)

AND of OR clauses

\[
\phi = (x_1 \lor \overline{x}_2) \land (x_2 \lor x_3 \lor x_4) \land (\overline{x}_1 \lor x_3 \lor \overline{x}_4)
\]

\( G(\phi) \)

\( k = 3 \)

\( x_1 = T \quad x_2 = T \quad x_4 = F \)

\( x_3 = T/F \)
Construction of $G(\phi)$

1. For each clause with $p$ literals, create a complete graph with $p$ vertices, with the literals as labels.

2. For any two vertices labeled $x_i$ and $\overline{x}_i$, connect them with an edge.

$k(\phi) =$ number of clauses

Claim: $\phi$ is satisfiable if and only if $G(\phi)$ has an independent set of size $k(\phi)$

Proof: ($\Rightarrow$) Say $\phi$ is satisfiable with some assignment $x_1, x_2, \ldots, x_n \in \{\text{True, False}\}$

Each clause must have at least one literal set to True.

$\rightarrow$ Take the corresponding vertex from each clause graph.

This set is an independent set because the edges across different clauses are of the kind $(x_i, \overline{x}_i)$ and both $x_i, \overline{x}_i$ could not have been true.

($\Leftarrow$) Say there is an independent set $S$ in $G(\phi)$ with size $k(\phi)$

$S$ must have exactly one vertex from each clause graph.

Set the corresponding literals to true.

There is doable because an independent set cannot have both $x_i$ and $\overline{x}_i$.

It gives us a satisfying assignment.
HW

1. \text{IND-SET} \leq \text{Vertex Cover}

2. \text{Vertex Cover} \leq \text{Feedback vertex set}