NP, NP-completeness
and
a Million Dollar Question

History of P and NP

- Notion of Efficient Algorithms there since ancient times

- Addition, Multiplication, GCD, Repeated squaring (Pingala), Astronomical calculations.

- [1950s] Dynamic Programming, Shortest Path, Simplex algorithm

- Doing better than brute force search

P (Polynomial time solvable)

- Edmonds [1965] proposed polynomial time as a characterization of efficient computation
  “It is by no means obvious whether or not there exists an algorithm whose difficulty increases only algebraically with the size of the graph”
  ”For practical purposes the difference between algebraic and exponential is more crucial than between finite and non-finite.”

- Why polynomial time?
  - if a procedure is considered efficient, running it n times might also be considered efficient.
  - Polynomial time remains independent of computation model.
  - Another perspective: if you double the input size, the running time gets multiplied by a constant.

Towards NP

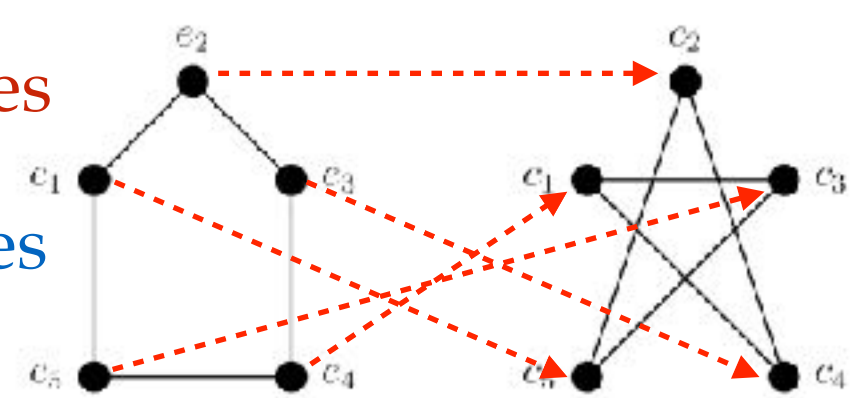
- [1960s] For many problems, people could not find better than exponential time algorithms.
  - Subset sum, Load balancing, Traveling Salesperson, Graphs Isomorphism, Primality, Linear programming, Minimum Circuit Size, Satisfiability
  - Not even for the decision versions:
    - e.g., is there a load allocation with makespan $\leq$ k
    - e.g., is there a matching of size $\geq$ r

Easily Verifiable Proofs

- Many problem which seemed hard have easily verifiable proofs for `yes' inputs.
- Load Balancing: is there a load allocation with makespan at most k?
  - Proof: an allocation with makespan k′ $\leq k$
  - Verifier: check if the proposed allocation is valid and its makespan
- Factorize Numbers: is a given number factorizable?
  - Proof: two numbers
  - Verifier: multiply the proposed two numbers and check if you get the input number.
- Not clear if 'no' inputs have easily verifiable proofs.

Easily Verifiable Proofs

- SAT: given a Boolean formula (CNF - AND of ORs), is there an assignment of variables, which makes it true?
  Example. $(\neg x \vee y) \wedge (\neg y \vee x)$
  - Proof: an satisfying assignment to the variables (example: True, False)
  - Verifier: check if the proposed assignment makes the formula true.
- Graph Isomorphism: given two graphs, are they isomorphic?
  - Proof: a mapping between two sets of vertices
  - Verifier: check if the given mapping preserves edges and non-edges



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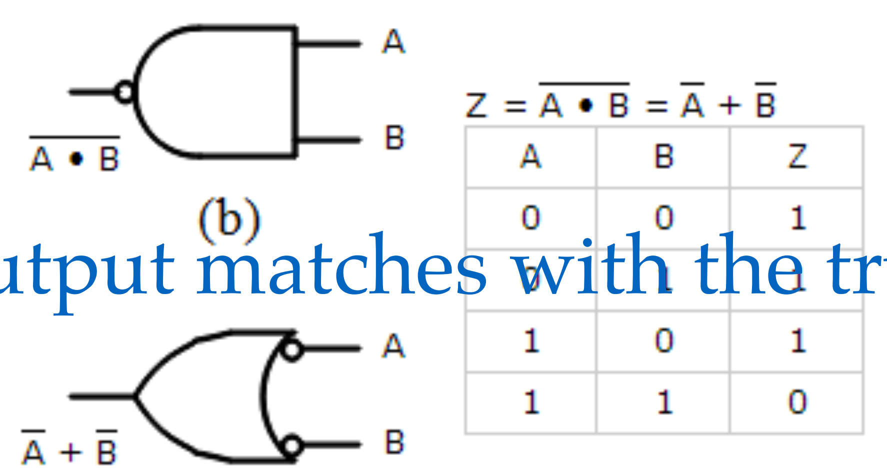
Easily Verifiable Proofs

- Subset Sum: given numbers $a_1, a_2, a_3, \ldots, a_n$ and a number b, is there a subset of a_i's that sum up to b ?
  - Proof: a subset of numbers
  - Verifier: check if the proposed subset has sum equal to b
- Circuit Size: given a Boolean function f truth table, is there a circuit with at most s gates that computes f ?
  - Proof: a circuit
  - Verifier: verify if the circuit output matches with the truth table for every input



A	B	Z
0	0	1
0	1	1
1	0	1
1	1	0

The Class NP

- Definition: a ‘yes or no’ decision problem is in NP if there is an easily verifiable proof for each `yes’ input.
- a ‘yes or no’ decision problem is in NP if there is a polynomial time Algorithm V (verifier), such that for any input x,
    - if x is a ‘yes’ input then there exists y s.t. $V(x,y) = \text{True}$.
    - if x is a ‘no’ input then for any y, $V(x,y) = \text{False}$

The Class NP

- P - can find the solution in polynomial time
  NP - can verify a proposed solution in polynomial time

- if a problem is in P, it is also in NP.

- A problem falling into NP is a positive thing.

- NP does not mean Non-Polynomial time.

Problems in NP?

- Given an integer n, are there integers x, y, z such that
  $x^3 + y^3 + z^3 = n$?
  e.g. for n = 39: $134476^3 - 159380^3 + 117367^3 = 39$.
  Not clear, because x, y, z can much larger than n. Efficient verification might not be possible.

- Given a graph, can we remove at most k edges to make it 3-colorable?
  Proof: k edges to remove and a coloring scheme with 3 colors.

- Given a graph, is there a matching of size at least k?
  Proof: a matching of size k, which verifier can check.
  Proof: verifiers ignores the proof and just finds the maximum matching. If at least k, then say yes.

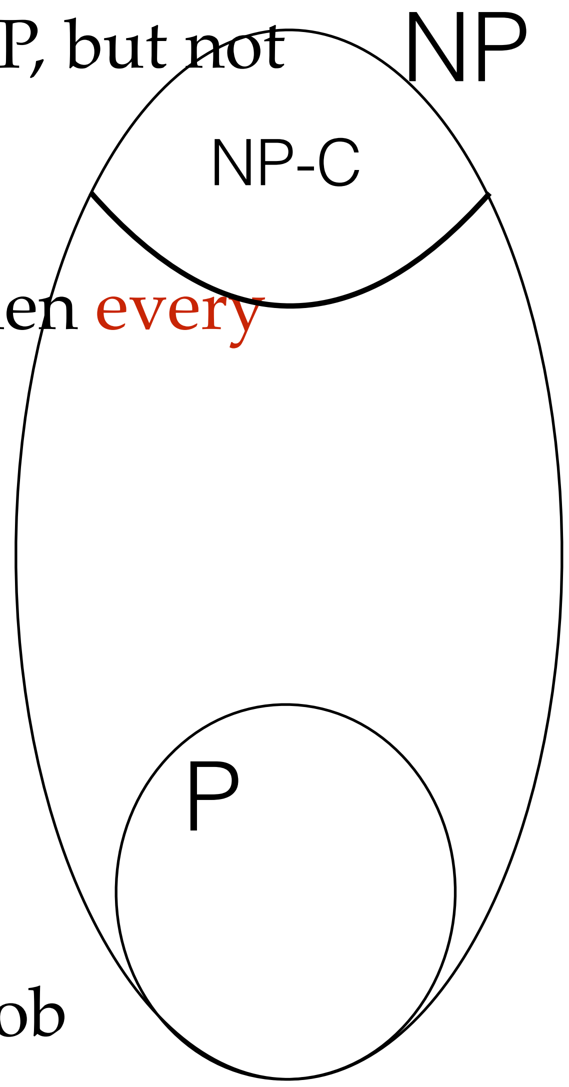
- Can you color the edges with red/blue, s.t. every subset of k vertices has edges with both colors?
  Not clear. If someone gives a coloring scheme, not clear how to verify it for every subset.

P vs NP

- Problems intuitively in class NP
  - is a given mathematical statement true?
    (a proposed proof can be verified)
  - is there a cure for a mentioned disease?
    (a proposed cure can be verified)
  - given the public key, can you find the private key?
    (a private-public key pair can be verified)
- P vs NP = Mechanical vs Creativity

NP-completeness

- Various problems like TSP, SAT were conjectured to be not in P, but not enough evidence.
- Cook-Levin [1971]: If SAT has a polynomial time algorithm then every problem in NP will have a polynomial time algorithm.
  - SAT is the hardest problem in NP
  - SAT is 'NP-complete'.
- Karp [1972]: 21 other problems are NP-complete.
  - TSP, Subset Sum, Integer Programming, Graph Coloring, Job Sequencing, Independent Set, 3D-matching etc.
  - They are all equivalent and are hardest problems in NP



Reductions

- Problem A reduces to problem B ($A \leq B$)
  - if A can be solved in polynomial time using a given subroutine that solves B.
  - task of solving A reduces to task of solving B
- Example: Taxi scheduling reduces to bipartite matching
- Example: Multiplication reduces to squaring
  - Multiplication is as easy as squaring
  - Squaring is as hard as Multiplication
- A reduces to B: 1) convert phi_A to phi_B. 2) Solution(phi_B) should be converted to solution(phi_A).
  Conclusion: A is as easy as B. B is as hard as A.
- $A \leq B$ and $B \leq C$ implies $A \leq C$

The tree of Reductions

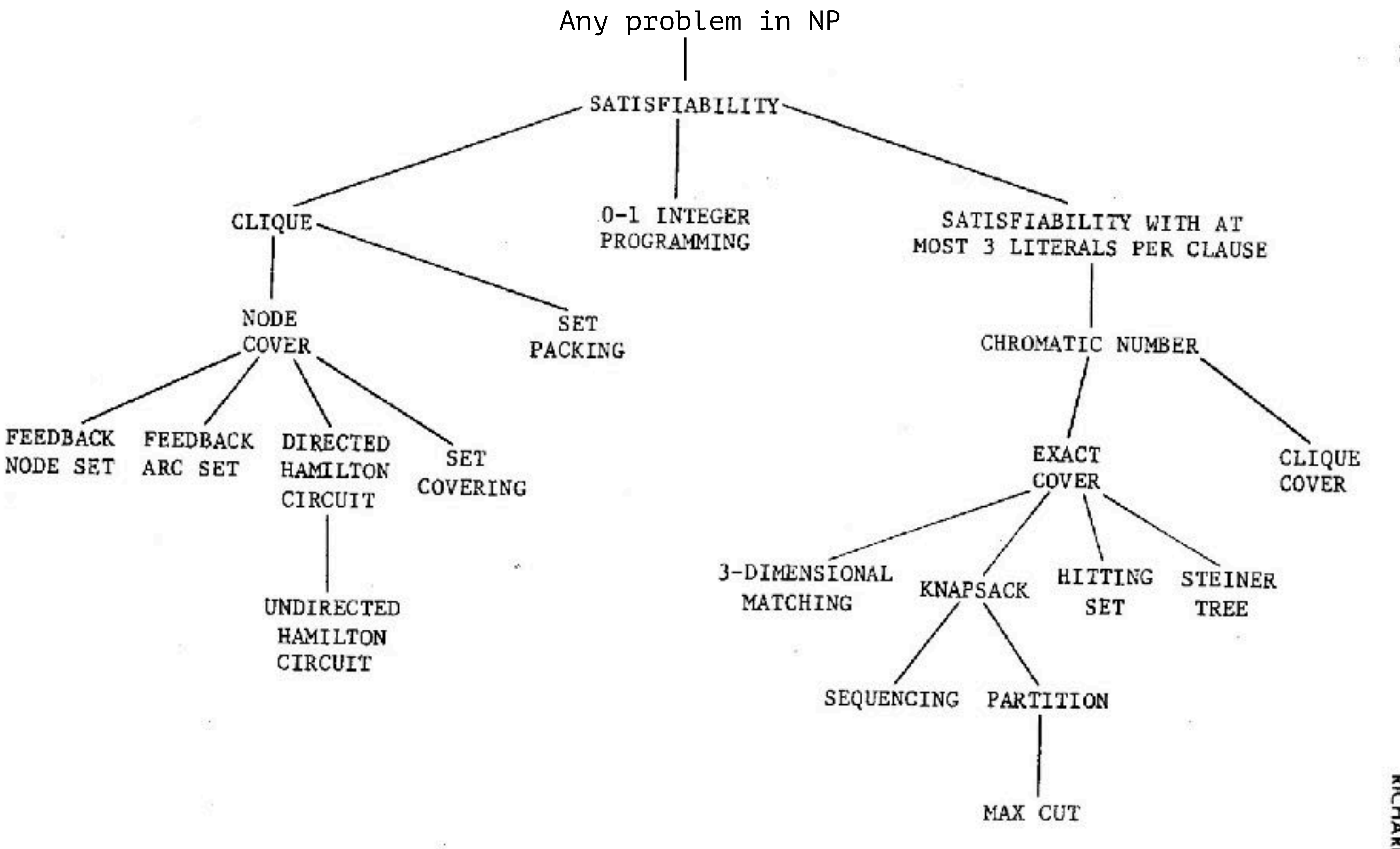
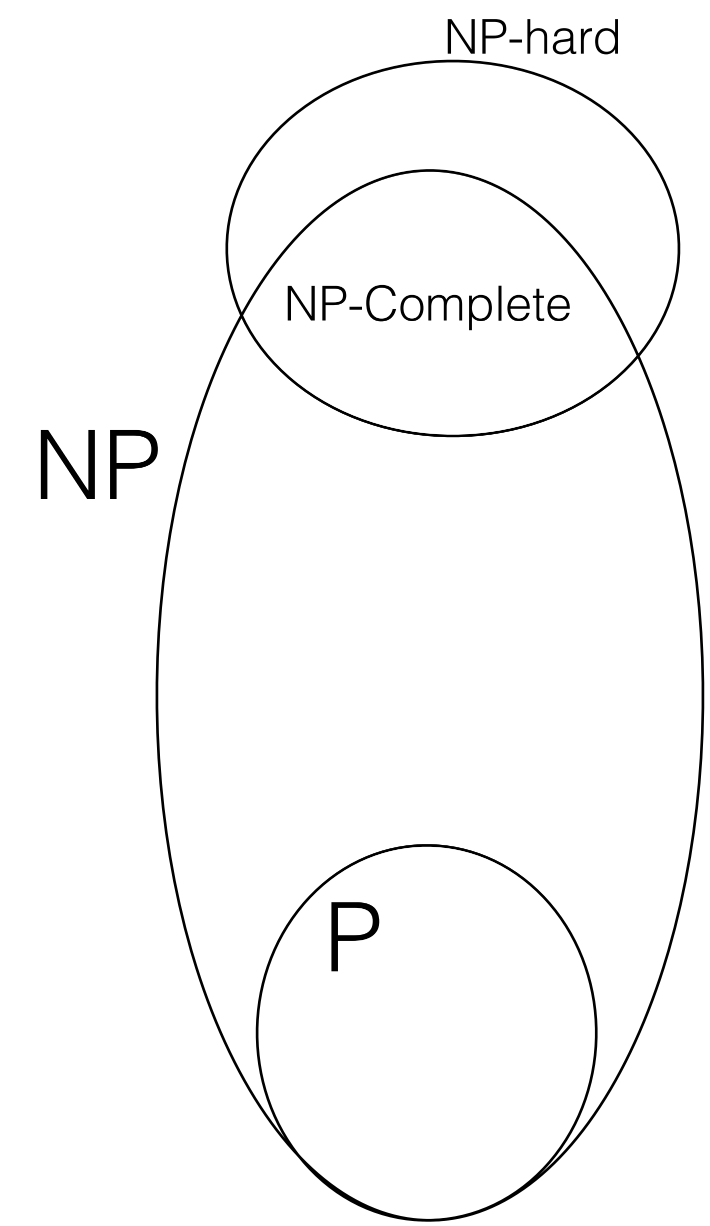


FIGURE 1 - Complete Problems

NP-complete and NP-hard

- Problem X is said to be NP-complete if
  1. X is in NP
  2. Every problem in NP reduces to X
- Problem Y is said to be NP-hard if
  - Every problem in NP reduces to Y



Summary

- Thousands of problems have been shown to be NP-complete.
- If you solve any of them, all of them get solved.
- One can say, there is just one NP-complete problem.
- People have not been able to give an efficient algorithm in last 50 years.
- P=NP would mean all these problems have efficient algorithms. And all diseases can be cured, all mathematical conjectures can be resolved, crypto systems can broken and so on..

Summary

- Widely believed $P \neq NP$, but no proof for it.
  Million Dollars for a proof either way.
- If you can't find a Polynomial time algorithm for a problem *X*, try to prove that it is NP-hard.
  - Choose a suitable NP-complete/NP-hard problem *H* and reduce *H* to your problem *X*.
  - I.e., *H* can be solved using a subroutine for *X*.
  - "I am not able to design an algorithm for it, but nobody could in last 50 years 😊"
- Most problems turn out to be either in P or NP-complete.
  - Exceptions: Graph Isomorphism, Minimum circuit Size

Babai 2015,
quasi-poly

3-colorability to SAT

- Example: 3-colorability reduces to SAT
- Given a graph, can we color vertices with 3 colors?
  - create Boolean variables to represent the `proof'
  - 3 Boolean variables for each vertex - $x_{i,}\, y_{i,}\, z_i$
  - encode the verification procedure as Boolean constraints
    - each vertex has a color — $(x_i \vee y_i \vee z_i)$ for each i
    - adjacent vertices have different colors — $\neg(x_i \wedge x_j)$, $\neg(y_i \wedge y_j)$, $\neg(z_i \wedge z_j)$ for every edge (i,j)
    - Boolean formula = AND of all the constraints.
  - Graph is 3-colorable if and only if there is an satisfying assignment for the above Boolean formula

Any problem in NP reduces to SAT [Section 8.4 in Kleinberg Tardos]

- There is a verifier algorithm V such that for any input x,
  - if x is a 'yes' input then there exists y s.t. $V(x,y)$ = True.
  - if x is a 'no' input then for all y, $V(x,y)$ = False
- Reduction: Given x, output a boolean formula $f(x)$ such that
  - if x is a 'yes' input then $f(x)$ has a satisfying assignment
  - if x in a 'no' input then $f(x)$ does not have a satisfying assignment
- Proof y encoded as Boolean variables.
- Each step of algorithm V will be converted to a Boolean constraint.

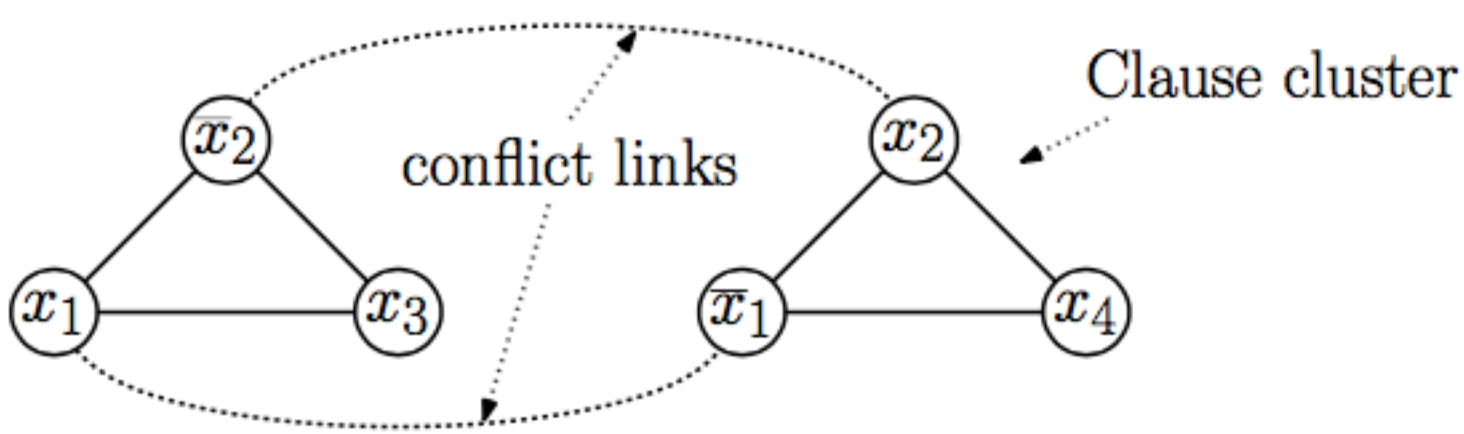
Any problem Q in NP reduces to SAT

- Algorithm V: Input $(1,0,1,0,0,..,\, y_1,\, y_2,\, \ldots,\, y_m)$
- Say it uses p bits memory and time T.
- Create another pT Boolean variables.
- At time t, an instruction will apply AND/OR/NOT on some memory locations and store it in another location
- $$z_{t+1,5} = z_{t,3} \vee z_{t,9}$$
- $f(x) =$ AND of all such Boolean constraints.
- $f(x)$ has a satisfying assignment (y,z)
  if and only if algorithm V outputs True on input $(x,\, y_1,\, y_2,\, \ldots,\, y_m)$
  if and only if x is a yes input.

SAT to IND-SET Reduction

- IND-SET: given a graph G and a number k, is there an independent set of size k?
- Reduction: Given a CNF formula ϕ, output a graph $G(\phi)$ and a number $k(\phi)$ such that
  - if ϕ has a satisfying assignment, then $G(\phi)$ has an independent set of size $k(\phi)$
  - if ϕ does not have a satisfying assignment, then $G(\phi)$ does not have any independent set of size $k(\phi)$
  - $\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$.
  - you have to choose one literal form every clause to be true and you can choose only one between x_2 and $\neg x_2$.

$G(\phi) =$



Thank you

References

- [1] https://math.stackexchange.com/questions/3141500/are-these-two-graphs-isomorphic-why-why-not
- [2] http://electronics-course.com/logic-gates