Lecture 1

1. Prove that various standard forms of linear programs are equivalent.

Form 1: maximize $w^T x$ subject to $Ax \leq b$
Form 2: minimize $w^T x$ subject to $Ax = b$, $x \geq 0$
Form 3: minimize $w^T x$ subject to $Ax \leq b$, $A'x = b'$, $A''x \geq b''$

2. Give an example of a set of constraints $Ax \leq b$ and two linear functions $w_1^T x$ and $w_2^T x$ such that $\max \{w_1^T x \mid Ax \leq b\}$ has a finite value while $\max \{w_2^T x \mid Ax \leq b\}$ is unbounded.

Lecture 2

1. Prove that the feasible region of a set of linear constraints, i.e., $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, is a convex set.

2. There were three claims made in class.
   (a) For any $w \in \mathbb{R}^n$ and a polyhedron $P$, the set of points $x \in P$ maximizing $w^T x$ forms a face.
   (b) Suppose $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and $z \in P$ is a point maximizing $w^T x$. Let $(A', b')$ be the subset of rows of $(A, b)$ which gives all the tight constraints for $z$, i.e., $A'z = b'$. Then, any point $y \in P$ satisfying $A' y = b'$ will also maximize $w^T x$ over $P$.
   (c) If $\alpha$ and $\beta$ are two points maximizing $w^T x$ in $P$. Then $\frac{\alpha + \beta}{2}$ will also maximize $w^T x$ in $P$.

We had proved Claim (b) in the class. First prove Claim (c) and then using (b) and (c), prove Claim (a).

Lecture 3

1. Three definitions of corners/vertices of a polyhedron were discussed in class.
   (a) Suppose $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. $z \in P$ is a vertex if there exists a subset of rows of $(A, b)$, say $(A', b')$, such that $A'z = b'$ and $\text{rank}(A') = n$. 

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(b) $z \in P$ is a vertex if there is no $y \in \mathbb{R}^n$ such that $z + y \in P$ and $z - y \in P$.

(c) $z \in P$ is a vertex if there exists $w \in \mathbb{R}^n$ such that $z$ is the unique point maximizing $w^T x$ over $P$.

Prove equivalence of these definitions which were not done in class, that is, (a) implies (c).

Lecture 4

1. For a set of points $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}^n$, let $\text{conv} (\alpha_1, \alpha_2, ..., \alpha_k)$ denote their convex hull, i.e.,
\[ \{ \sum_{i=1}^{k} \lambda_i \alpha_i \mid \sum_{i=1}^{k} \lambda_i = 1, \ 0 \leq \lambda_i \forall i \} \]. Prove the following.
\[
p \in \text{conv}(p_1, p_2), \ q \in \text{conv}(q_1, q_2), \ r \in \text{conv}(p, q) \implies r \in \text{conv}(p_1, p_2, q_1, q_2)
\]

2. Fourier Motzkin Elimination: Consider the following two system of inequalities.
\[
\begin{align*}
x_1 &= \lambda + 2(1 - \lambda) \\
x_2 &= \lambda - (1 - \lambda) \\
0 &\leq \lambda \leq 1
\end{align*}
\]
\[
\begin{align*}
2 - x_1 &= (x_2 + 1)/2 \\
0 &\leq 2 - x_1 \leq 1
\end{align*}
\]

When you apply Fourier Motzkin Elimination (FME) on the left system, you get the right one. Show that

(a) for any point $(x_1, x_2, \lambda)$ satisfying the left system, the point $(x_1, x_2)$ must satisfy the right system.

(b) for any point $(x_1, x_2)$ satisfying the right system, there must exist $\lambda \in \mathbb{R}$ such that $(x_1, x_2, \lambda)$ satisfies the left system.

Argue that these also hold when FME is applied on a general system of linear inequalities.

3. Optimization: Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $w \in \mathbb{R}^n$, find the value $\max w^T x$ subject to $Ax \leq b$.

Feasibility: Given $C \in \mathbb{R}^{m \times n}$, $d \in \mathbb{R}^m$, decide (yes or no) if there exists a point $x \in \mathbb{R}^n$ satisfying $Cx \leq d$.

Show that Optimization reduces to Feasibility. That is, you can solve the optimization question if you are allowed to use the feasibility subroutine polynomially many times.