# CS602: Homework Problems

January 23, 2020

## Lecture 1

1. Prove that various standard forms of linear programs are equivalent.

Form 1: maximize $w^{\intercal}x$ subject to $Ax \leq b$

Form 2: minimize $w^{\intercal}x$ subject to $Ax = b$, $x \geq 0$

Form 3: minimize $w^{\intercal}x$ subject to $Ax \leq b$, $A'x = b'$, $A''x \geq b''$

2. Give an example of a set of constraints $Ax \leq b$ and two linear functions $w_1^{\intercal}x$ and $w_2^{\intercal}x$ such that $\max\{w_1^{\intercal}x \mid Ax \leq b\}$ has a finite value while $\max\{w_2^{\intercal}x \mid Ax \leq b\}$ is unbounded.

## Lecture 2

1. Prove that the feasible region of a set of linear constraints, i.e., $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, is a convex set.

2. There were three claims made in class.

   (a) For any $w \in \mathbb{R}^n$ and a polyhedron $P$, the set of points $x \in P$ maximizing $w^{\intercal}x$ forms a face.

   (b) Suppose $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and $z \in P$ is a point maximizing $w^{\intercal}x$. Let $(A', b')$ be the subset of rows of $(A, b)$ which gives all the tight constraints for $z$, i.e., $A'z = b'$. Then, any point $y \in P$ satisfying $A'y = b'$ will also maximize $w^{\intercal}x$ over $P$.

   (c) If $\alpha$ and $\beta$ are two points maximizing $w^{\intercal}x$ in $P$. Then $\frac{\alpha+\beta}{2}$ will also maximize $w^{\intercal}x$ in $P$.

   We had proved Claim (b) in the class. First prove Claim (c) and then using (b) and (c), prove Claim (a).

## Lecture 3

1. Three definitions of corners/vertices of a polyhedron were discussed in class.

   (a) Suppose $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. $z \in P$ is a vertex if there exists a subset of rows of $(A, b)$, say $(A', b')$, such that $A'z = b'$ and $\text{rank}(A') = n$.

(b) $z \in P$ is a vertex if there is no $y \in \mathbb{R}^n$ such that $z + y \in P$ and $z - y \in P$.

(c) $z \in P$ is a vertex if there exists $w \in \mathbb{R}^n$ such that $z$ is the unique point maximizing $w^\intercal x$ over $P$.

Prove equivalence of these definitions which were not done in class, that is, (a) implies (c).

# Lecture 4

1. For a set of points $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}^n$, let $\mathrm{conv}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ denote their convex hull, i.e., $\{\sum_{i=1}^k \lambda_i \alpha_i \mid \sum_{i=1}^k \lambda_i = 1,\ 0 \le \lambda_i\ \forall i\}$. Prove the following.

$$p \in \mathrm{conv}(p_1, p_2),\ q \in \mathrm{conv}(q_1, q_2),\ r \in \mathrm{conv}(p, q) \implies r \in \mathrm{conv}(p_1, p_2, q_1, q_2)$$

2. **Fourier Motzkin Elimination**: Consider the following two system of inequalities.

$$\begin{aligned} x_1 &= \lambda + 2(1-\lambda) \\ x_2 &= \lambda - (1-\lambda) \\ 0 &\le \lambda \le 1 \end{aligned} \qquad\qquad \begin{aligned} 2 - x_1 &= (x_2+1)/2 \\ 0 &\le 2 - x_1 \le 1 \end{aligned}$$

When you apply Fourier Motzkin Elimination (FME) on the left system, you get the right one. Show that

(a) for any point $(x_1, x_2, \lambda)$ satisfying the leftt system, the point $(x_1, x_2)$ must satisfy the right system.

(b) for any point $(x_1, x_2)$ satisfying the right system, there must exist $\lambda \in \mathbb{R}$ such that $(x_1, x_2, \lambda)$ satisfies the left system.

Argue that these also hold when FME is applied on a general system of linear inequalities.

3. **Optimization:** Given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $w \in \mathbb{R}^n$, find the value $\max w^\intercal x$ subject to $Ax \le b$.

**Feasibility:** Given $C \in \mathbb{R}^{m\times n}$, $d \in \mathbb{R}^m$, decide (yes or no) if there exists a point $x \in \mathbb{R}^n$ satisfying $Cx \le d$.

Show that Optimization reduces to Feasibility. That is, you can solve the optimization question if you are allowed to use the feasibility subroutine polynomially many times.