CS602: Homework Problems

February 3, 2020

Lecture 6

Que 1 [Farkas' Lemma Different Forms]. Prvoe that the three forms of Farkas' lemma can be derived from one another. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$.

- If $Ax \leq b$ is not feasible then there exists $y \in \mathbb{R}^m$ with $y \geq 0$ such that $y^\mathsf{T} A = 0$, but $y^\mathsf{T} b = -1$.
- If $Ax = b, x \geq 0$ is not feasible then there exists $y \in \mathbb{R}^m$ such that $y^\mathsf{T} A \geq 0$, but $y^\mathsf{T} b = -1$.
- If $Ax \leq b, x \geq 0$ is not feasible then there exists $y \in \mathbb{R}^m$ with $y \geq 0$ such that $y^\mathsf{T} A \geq 0$, but $y^\mathsf{T} b = -1$.

Que 2 [Farkas' Lemma via FM elimination]. An alternate proof of the Farkas' Lemma was suggested in the class using Fourier Motzkin elimination. That seems like an excellent idea. It gives a shorter proof. You need to prove two things for this.

- If you start with a infeasible system and eliminate a variable using FM, then the new system is still infeasible.
- The new constraints generated in the process are always non-negative combinations of '$\leq$' constraints + arbitrary real combinations of '$=$' constraints.

Once you prove these two statements you argue as follows: continue eliminating variables one by one. When you are reach a single variable (say x_1), the system must still be infeasible. How does that look like? For infeasibility, it must contain two constraints like

$$x_1 \leq \alpha, x_1 \geq \beta \text{ for some } \alpha < \beta, \text{ or}$$

$$x_1 = \alpha, x_1 \geq \beta \text{ for some } \alpha < \beta, \text{ or}$$
$$x_1 = \alpha, x_1 = \beta \text{ for some } \alpha \neq \beta$$

In all three cases, you can combine the pair of constraints to get $0 \leq -1$.

Remark: Recall that to prove the corollary about the polyhedral cone, we anyways used FME. And then, showed how the corollary implies Farkas' Lemma. Instead, it would be much simpler to directly use FME to prove Farkas' Lemma (as described above).

Homework 1. *Given $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$, prove that the following set (polyhedral cone) is convex:*

$$C = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_k v_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \geq 0.\}$$

Recall that a set is convex if for any two points in the set the line joining them is also in the set.

Homework 2 (Separating Hyperplane Theorem)**.** *Let $P \subseteq \mathbb{R}^n$ be a convex (and closed) body. And there is a point y such that $y \notin P$. Then there exists a hyperplane separating them. That is, there exists $\beta \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$ such that*

$$\beta^\mathsf{T} y > \delta, \text{ and} \tag{1}$$

$$\beta^\mathsf{T} p \leq \delta \text{ for each } p \in P. \tag{2}$$

You can prove this by taking the point $z \in P$ that is closest to y. Take the hyperplane passing through z which is orthogonal to $y - z$. That is, $\beta = y - z$ and $\delta = (y - z)^\mathsf{T} z$.

Lecture 7: LP Duality

Homework 3. *Consider the following linear program.*

$$\begin{aligned} \max\ 2x_2 - x_1 & \qquad \textit{subject to} \\ 2x_1 - x_2 &\leq 0 \\ 2x_1 + x_2 &\leq 5 \\ x_1 + 3x_2 &\leq 10 \\ x_2 - x_1 &\leq 6 \\ -x_1 - 2x_2 &\leq 0 \end{aligned}$$

Find the optimal value w^* and one of the feasible points that achieves optimal value. What is the positive combination of the constraints that will give you $x_1 - 2x_2 \leq w^*$.

Theorem 1 (Strong Duality)**.** *$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$, $w \in \mathbb{R}^n$. Let $w^* = \max\{w^\mathsf{T} x \mid Ax \leq b\}$. Then, there exists $y \in \mathbb{R}^m$ such that*

$$\begin{aligned} y &\geq 0 \\ y^\mathsf{T} A &= w^\mathsf{T} \\ y^\mathsf{T} b &= w^* \end{aligned}$$

Homework 4 (Only if you are interested)**.** *Show strong duality using Fourier Motzkin elimination. One way to try: Take the LP in Homework 3, add a constraint $x_3 = 2x_2 - x_1$ for the optimizing function. Now, eliminate x_1, x_2 and you will left with constraints of the form $x_3 \leq \gamma$ and $x_3 \geq \delta$. Does one of these give you the optimal value?*

Homework 5. *Prove the strong duality theorem using Claim 2 below about polyhedral cones seen in the previous class.*

Claim 2 (Separating Hyperplane for a Cone)**.** *Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ and $C = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_k v_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \geq 0\}$ be the cone generated by them. Suppose there is a vector $u \in \mathbb{R}^n$ which is not in the cone C. Then, there must exists $\beta \in \mathbb{R}^n$ such that*

$$\begin{aligned} \beta^\mathsf{T} u &> 0, \text{ and} \\ \beta^\mathsf{T} v_i &\leq 0 \text{ for each } 1 \leq i \leq k. \end{aligned}$$

Lecture 8: Various forms of LP Duality

No.	Primal LP	Dual LP
(a)	$\max w^\mathsf{T} x : Ax \leq b$	$\min b^\mathsf{T} y : y \geq 0,\ A^\mathsf{T} y = w$
(b)	$\max w^\mathsf{T} x : Ax \leq b,\ x \geq 0$	$\min b^\mathsf{T} y : y \geq 0,\ A^\mathsf{T} y \geq w$
(c)	$\max w^\mathsf{T} x : Ax = b,\ x \geq 0$	$\min b^\mathsf{T} y : A^\mathsf{T} y \geq w$

Homework 6 (Duality in different forms)**.** *We had proved that primal optimal is equal to the dual optimal for the LP pair (a) in the above table. Prove this primal-dual equality for (b) and (c) too (using (a)).*

Homework 7 (Dual of Dual)**.** *Prove that when you take dual of the dual then you get back the original LP. To prove this, take one of the three forms on the left hand side of the table, say* LP_1*. Take its dual* LP_1^* *as given in the table on its right hand side.* LP_1^* *is a minimization problem. Express it as a maximization problem using appropriate sign changes, say* $\bar{LP}_1^*$*. You will get an LP which matches one of the three forms in the left hand side. Now, you write the dual LP for this (using the table), which is a minimization problem, say* $(\bar{LP}_1^*)^*$*. Now, again do an appropriate sign change on* $(\bar{LP}_1^*)^*$ *to get a maximization problem. Did you get the original LP?*