CS602: Homework Problems

February 20, 2020

Lecture 10

Homework 1. *We had shown in the class the LP we write for bipartite maximum matching always has an integral optimum. Now, suppose each edge has a weight given by* $\{w_e\}_{e\in E}$ *and we are interested in maximum weight matching. Naturally, we will keep the same constraints but change the optimizing function to* $\max \sum_e w_e x_e$. *Can you again show that for any arbitrary optimizing function, there is always an integral optimum solution?*

Homework 2. *We wrote the following dual linear program for the maximum matching LP in bipartite graphs. Let the graph be* $G(V, E)$.

$$\begin{aligned} \min \quad & \textstyle\sum_{v\in V} y_v \quad \textit{subject to} \\ \textit{for each } v \in V, \quad & y_v \geq 0, \\ \textit{for each } e = (u, v) \in E, \quad & y_u + y_v \geq 1. \end{aligned}$$

This looks like minimum vertex cover problem. Prove that in case of bipartite graphs, there is always an integral optimal solution for this LP. Argue using LP duality that maximum matching and minimum vertex cover have same sizes in bipartite graphs.

Homework 3. *For general graphs, use a direct combinatorial argument (i.e., without LPs) to show that maximum matching size is at most the minimum vertex cover size.*

Homework 4. *Consider the bipartite matching LP.*

$$\textit{for each } e \in E, \quad x_e \geq 0 \tag{1}$$

$$\textit{for each } v \in V, \quad \textstyle\sum_{e\in\delta(v)} x_e \leq 1. \tag{2}$$

Recall that for a matching $M \subseteq E$, *the corrsponding matching point* $\chi^M \in \{0, 1\}^E$ *is defined as*

$$\chi_e^M = \begin{cases} 1 & \textit{if } e \in M \\ 0 & \textit{otherwise.} \end{cases}$$

Show that every point $\alpha \in \mathbb{R}^E$ *that satisfies (4) and (5) can be written as convex combination of matching points. In some sense, we have already shown this in the class, but you need to connect all the dots.*

Lecture 11

Homework 5. *Can you prove Menger's theorem using LP duality (max no. of edge-disjoint s-t paths is equal to min (s,t)-cut)? Another way to prove this is via using the bipartite matching theorem (in a bipartite graph, max matching size is equal to min vertex cover size).*

Homework 6. *Can you prove Max flow Min cut theorem using Menger's theorem. How to do it: Try to view Menger's theorem as just Max flow min cut when all edge capacities are 1. Given a flow network with arbitrary edge capacities, can you construct another equivalent flow network where edge capacities are all 1?*

Lecture 12

Homework 7. *Let M be a matching in a graph G. If M is not a maximum size matching then prove that there exists an augmenting path in G with respect to M.*

Homework 8. *Given a matching in a bipartite graph, can you give an efficient algorithm to find an augmenting path (or report that it does not exist)?*

Homework 9. *Consider the LP for minimum weight perfect matching in a bipartite graph.*

$$\min \quad \textstyle\sum_{e\in E} w_e x_e \quad \textit{subject to} \tag{3}$$
$$\textit{for each } e \in E, \quad x_e \geq 0 \tag{4}$$
$$\textit{for each } v \in V, \quad \textstyle\sum_{e\in\delta(v)} x_e = 1. \tag{5}$$

Show that there is always an optimal integral solution. In other words, integral optimal value is same as the fractional optimal value. As we had done in a class, start with a fractional feasible solution and convert it into an integral feasible solution such that it has a smaller or same value.

Lecture 13

Homework 10. *Recall the primal dual scheme we saw in the class for min weight bipartite matching. We can also design a slightly different scheme for improving the dual, where we don't need to worry about half-integral ϵ. Do the following: Let U be the minimum vertex cover and let $U_L \subseteq U$ be the set of left vertices in the vertex cover. Let $U'_R \subseteq V \setminus U$ be the set of right vertices not in the vertex cover.*

The scheme is as follows: decrease U_L values by ϵ and increase U'_R values by ϵ. Argue that

- *the new dual solution is still feasible. That is, if any tight edge sees $+\epsilon$ at one end, it must see $-\epsilon$ at the other end.*
- *$|U'_R| > |U_L|$. Thus, the total dual value will strictly increase.*
- *If the initial weights are integral then ϵ is always integral throughout the algorithm.*

Homework 11. *Argue that the augmenting path algorithm for bipartite graphs not only gives you maximum matching, but also minimum vertex cover. Here is how to argue: the algorithms stops* *when it can find no more augmenting paths. You have some free vertices on the left and some free vertices on the right. Since there are no augmenting paths, the right free vertices are not reachable from left free vertices via alternating paths. Consider the BFS tree with alternating matching and non-matching edges that starts with free vertices on the left. Stare at it and see if you can find a vertex cover. Just find one vertex cover whose size is same as maximum matching. Then you don't need to prove optimality.*