CS602: Homework Problems

March 26, 2020

Lecture 14

Homework 1. *Show that the following two problems are equivalent.*

Hitting set: *Let U be a set with each element having a cost. Given a collection of subsets $S_1, S_2, \dots, S_k \subseteq U$, the goal is to find a set $T \subseteq U$ of minimum cost such that for each $1 \le i \le k$, $T \cap S_i \neq \emptyset$.*

Set cover: *Let U be a set of elements. Given a collection of subsets $S_1, S_2, \dots, S_k \subseteq U$ each having a cost, the goal is to find a sub-collection $S_{i_1}, S_{i_2}, \dots, S_{i_p}$ ($1 \le i_1 < i_2 < \cdots < i_p \le k$) of minimum cost such that $S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_p} = U$.*

Homework 2. *Let $G(V, E)$ be a graph with edge costs. Suppose $T \subseteq E$ is a set of edges such that for every subset $S \subseteq V$ of vertices, T has at least one edge from the cut $E(S, \overline{S})$. Argue that (V, T) forms a connected graph. Moreover, if we are asking for such a set T of minimum cost, then it must be a spanning tree.*

Homework 3. *In the above question, we are trying to express minimum spanning tree as a hitting-set problem. Can you express the Steiner tree problem as a hitting-set problem?*

Lecture 15

Homework 4. *Let $G(V, E)$ be a graph with a set of terminals $T \subseteq V$. Suppose $F \subseteq E$ is a set of edges such that for every subset $S \subseteq V$ of vertices with $S \cap T \neq \emptyset$ and $\overline{S} \cap T \neq \emptyset$, F has at least one edge from the cut $\delta(S)$. Argue that (V, F) forms a graph, where all terminal vertices are connected with each other.*

Homework 5. ***Approximation Algorithm for Steiner Tree:*** *Given grpah G and set of terminals $T \subseteq V$, build a new complete graph $H(T, F)$ on the terminal vertices where an edge (t_1, t_2) has weight equal to the distance between t_1 and t_2 in G. Compute a minimum weight spanning tree in H and then take the union of shortest paths in G that corresponds to edges in this spanning tree. Break cycles if any (will there be any?).*

Prove that this is a 2-approximation algorithm for graphical Steiner tree. Also, show this for Euclidean Steiner tree (distance between any pair is the Euclidean distance).

In fact, for Euclidean Steiner tree, the approximation ratio is conjectured to be $2/\sqrt{3}$. You should think about this.

Homework 6. *Can you try to express Eulidean Steiner tree problem as an integer program? (I don't know whether it is possible).*

Lecture 16 and 17

Homework 7. *We had seen the primal-dual algorithm for Steiner tree/forest in the class. Let x be the final primal solution after the pruning step. Suppose the primal and the dual approximately satisfy the complementary slackness condition with approximation factor γ,*

$$y_S \neq 0 \implies \sum_{e \in \delta(S)} x_e \leq \gamma.$$

That is, the number of edges of the primal solution that intersect with the cut-set $\delta(S)$ is at most γ. Show that it implies that the cost of x is at most γ times the cost of optimal Steiner tree/forest.

Homework 8. *Give an example where γ turns out to be more than 2.*

Homework 9. *Consider the Steiner tree instance where every vertex is a terminal and you want to connect each vertex with every other vertex. Basically, we want a minimum spanning tree. If you run the primal-dual algorithm for minimum spanning tree, do you get the exact optimal solution?*

Similarly, consider an instance where you have only two terminals. Basically we want a shortest path between s and t. Show that the primal-dual algorithm gives an exact optimal solution here. Basically show that $\gamma = 1$ after the pruning step.

Let x be the primal **integral** solution output by the algorithm. Let y be the dual solution (not necessarily integral) output by the algorithm. Let x_I^* be the primal integral optimal solution. Let f be the primal function to be minimized and let g be the dual function to maximized.

Homework 10. *Suppose one can prove that*

$$f(x) \leq \alpha \times g(y).$$

Then show that

$$f(x) \leq \alpha \times f(x_I^*).$$

Homework 11. *Was the pruning step really important for the 2-approximation bound?*

Our analysis showed that the said algorithm gives a 2-approximate solution. Can there be a different analysis of the same algorithm that gives a better approximation factor? The answer is no, as long as you stick to the primal-dual analysis. The following tells you why.

Let $G(V, E)$ be a graph with edge costs $\{c_e\}$. Suppose we want to build a minimum cost spanning tree. We wrote the following LP relaxation.

$$\begin{array}{rcl} \min & \sum_{e \in E} c_e x_e & \text{subject to} \\ \text{for each } e \in E, & x_e \geq 0, & \\ \text{for each non-empty } S \subseteq V, & \sum_{e \in \delta(S)} \geq 1. & \end{array}$$

Homework 12. *Construct a graph G where for the above LP, the cost of integral optimal solution is **almost** twice of the cost of optimal solution. More specifically, for $|V| = n$,*

$$\textit{Cost-of-integral-optimal} = (2 - 2/n) \times \textit{Cost-of-optimal}$$

The above factor of $2 - 2/n$ is known as **Integrality Gap**.

Homework 13. *Now, argue for the same example that if x is an integral primal solution and y is a dual solution then*

$$f(x) \geq (2 - 2/n) \times g(y).$$

And thus, the primal-dual analysis can at best give an approximation factor of $2 - 2/n$ (when we took n terminals).

Homework 14. *Consider two generalizations: (i) suppose each pair of terminals (s_i, t_i) comes with a number p_i and we want to have p_i edge disjoint paths between s_i and t_i. (ii) suppose instead of pairs we have a collection of subsets of terminals, say, $S_1, S_2, \ldots, S_t \subseteq V$. We want a subgraph where each S_i is connected within (we don't care about connectivity of S_i with S_j).*

Can you write ILP and LP relaxations for these two problems. Do you think the primal-dual algorithm will work for these as well?

Lecture 18

Homework 15. *Recall the LP we had written for Steiner Tree/Forest. It had exponentially many constraints. Can you design a polynomial-time separation oracle for it. That is, given an arbitrary point $x \in \mathbb{R}^E$ (not necessarily integral), you need to decide whether x satisfies all the constraints in the LP. And if not, find a violated constraint. Recall that for an integral point, checking the constraints was simple (just look at the connected components), but for an arbitrary point it will not be as straighforward.*

Hint: *Note that (s, t)-mincut problem can be solved in polynomial time.*