CS602: Homework Problems

March 28, 2020

Lecture 19

Linearity of Expectation

1. Write a proof of linearity of expectation.
2. If you toss an unbiased coin n times, what is the expected number of heads?
3. For a sequence on n independent coin tosses, what is the expected number of two consecutive heads?
4. Suppose you put m balls into n bins randomly and independently, that is, for each ball the probability that it falls into a particular bin is $1/n$. What is the expected number of empty bins?
5. There are 48 students in our class. Suppose each student has a uniformly random birthday, i.e., the probability that any particular date is the student's birthday is $1/365$. What is the expected number of pairs of students who have a common birthday.
6. Suppose you have a random number generator that gives an integer between 0 and 9, each with probability $1/10$. What is the expected number of trials till you see each of the ten numbers at least once.

Independence: Suppose you choose a random integer x between 1 and 60. Which of the following events are dependent or independent of each other? First try to answer intuitively and then see what the calculations say.

- x is divisible by 4.
- x is divisible by 5.
- x is divisible by 6.

Maximum Satisfiability

- In the lecture, we saw some randomized algorithms with certain approximation guarantees. We also discussed some ideas on how to convert them into a deterministic algorithm. Write those deterministic algorithms formally and prove that the same approximation guarantees hold for them.

- Complete the analysis of the (3/4)-approximation algorithm.
- Consider the the Max SAT instance with four clauses $(x_1 \vee x_2) \wedge (x_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2)$, and each clause having weight 1. Prove that the integrality gap for the LP we wrote is 3/4. That is, the LP optimal value is 4 and the actual optimal value is 3.

Max Directed Cut: We want to find a maximum weight directed cut. That is, given a directed graph with edge weights ($w_{i,j}$ for the edge going from i to j), we want to find a subset of vertices U so that the sum $\sum_{i \in U,\ j \notin U} w_{i,j}$ is maximized. Write an integer program for this problem.