CS602: Discussion and Homework Problems

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Lecture 5

There were some questions which we left unanswered in the class. Let us recall these.

Notation. m is the number of constraints, n is the number of variables. The entries of A, b are all integers, and ℓ is the upper limit on their number of bits.

Bit complexity of the optimal value. When we reduced optimization to feasibility using binary search, someone asked what should be the stopping criterion? It can be the following: stop when you find w^* such that

$$Ax \leq b, \ w^{\mathsf{T}}x \geq w^* \text{ is feasible,}$$

$$\text{and } Ax \leq b, \ w^{\mathsf{T}}x \geq w^* + \epsilon \text{ is not feasible.}$$

Then you know the optimal value is between w^* and $w^* + \epsilon$. But, this tells us only an approximate value of of the optimal solution. How do you get the exact value? Here is what one needs to prove.

Claim 1. *Let $OPT = \max\{w^{\mathsf{T}}x \mid Ax \leq b\}$. Then the value OPT can be written as a rational number p/q, where both p, q are integers with $\text{poly}(\ell, m, n)$ many bits.*

If the above claim is true then we can safely choose ϵ to be $1/2^{\text{poly}(\ell,m,n)}$. This will ensure that we get the exact value of OPT. We will come to the proof later.

Exponential time algorithm for feasibility. We discussed an exponential time algorithm to decide feasibility of $Ax \leq b$. We said if the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ has a vertex, then that vertex is given by the unique solution of $A'x = b'$ for some subset (A', b') of rows of A, b, where A' has n linearly independent rows. Thus, to look for a vertex, we can go over all possible choices of A' and see if $x = A'^{-1}b'$ satisfies all inequalities $Ax \leq b$.

What if P has no vertex? This is kind of a degenerate scenario, but we still need to handle this. One situation where this might happen is when $m < n$, but there can also be examples, where $m \geq n$, but still no vertex. Note that P still might have edges, or higher dimensional faces.

Consider the following example.

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_1 + x_2 &\geq 0 \\ x_3 &\leq 1 \\ x_3 &\geq 0 \end{aligned}$$

Note that in this example, $n = 3$ and if we take any subset of 3 inequalities, they won't be linearly independent (on the variable side). This means that this polyhedron does not have vertices. Note that it has four edges which are infinitely long. Does this always happen when there are no vertices?

There can be multiple ways to deal with this situation. One is to reduce the number of variables somehow. Another way is the following claim.

Claim 2. *Let $Ax \leq b$ have some feasible solution, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then there exists a subset (A', b') of rows of (A, b) such that **every** solution y satisfying $A'y = b'$ also satisfies $Ay \leq b$ (i.e., y is feasible).*

Remark: Note that when $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ has a vertex then the claim is obvious, because the vertex is the **unique** solution of $A'y = b'$ for some (A', b') with $\text{rank}(A') = n$. The claim is essentially saying that when there is no vertex, then we have some edge or face that is unbounded (infinitely long). How do we prove this claim?

Proof of Claim 2. Let us write the m constraints given by $Ax \leq b$ as

$$a_i^{\mathsf{T}} x \leq b_i \text{ for } 1 \leq i \leq m.$$

Let α be a feasible point. Some of the constraints might be tight for α. Let for some $k \in \{0, 1, 2, \ldots, m\}$, we have that

$$a_i^{\mathsf{T}} \alpha = b_i \text{ for } 1 \leq i \leq k, \tag{1}$$

$$a_i^{\mathsf{T}} \alpha < b_i \text{ for } k+1 \leq i \leq m. \tag{2}$$

If there no tight constraint, we take $k = 0$.

We will try to move around α to get a feasible point with more tight constraints. Consider a direction $\delta \in \mathbb{R}^n$ such that

$$a_i^{\mathsf{T}} \delta = 0 \text{ for } 1 \leq i \leq k. \tag{3}$$

(Can you always find such a δ?) Now, if we move along δ, that is, go to a point $\beta = \alpha + \delta$, then β still satisfies the k tight constraints i.e.,

$$a_i^{\mathsf{T}} \beta = b_i \text{ for } 1 \leq i \leq k. \tag{4}$$

We ask whether β is still feasible, i.e., whether $A\beta \leq b$?

Case 1: Suppose for all choices of δ satisfying (3), we have that $\beta = \alpha + \delta$ satisfies $A\beta \leq b$. This just means that every point β satisfying (4) satisfies $A\beta \leq b$. This proves the claim.

Case 2: Suppose there exists a direction δ that satisfies (3) but, $\beta = \alpha + \delta$ goes outside the feasible region, i.e., it violates one or more of the rest of the inequalities. We know that α is feasible. Take $0 < \epsilon < 1$ to be the largest value so that $\alpha + \epsilon\delta$ is still feasible. Why does such ϵ exist?.

Because $\alpha + \epsilon\delta$ anyways satisfies the tight constraints ($a_i^\intercal(\alpha + \epsilon\delta) = b_i$ for $1 \le i \le k$) for any ϵ. And non-tight constraints (2) give you some room to choose a nonzero ϵ. Since we take the largest possible ϵ, it must happen that some new constraint becomes tight for $\alpha + \epsilon\delta$, say,

$$a_{k+1}^\intercal(\alpha + \epsilon\delta) = b_{k+1}.$$

Thus, we have a new point that is feasible but now satisfies $k + 1$ tight constraints. Now, go back and run the same argument again with the new point.

We have shown that either we can add a new tight constraint (case 2), or we are in a situation that **every** point satisfying the current set of tight constraints is feasible (case 1). This proves the claim.

□

Homework Problem 1. With Claim 2, can you now give an exponential time algorithm to check feasibility?

Feasibility in NP. We were arguing that the feasibility problem is in NP. For that we needed that there is always a solution with small enough bit complexity.

Homework Problem 2. With Claim 2, can you show that if $Ax \le b$ is feasible then it has a feasible solution with rational numbers, where numerators and denominators both have only $\text{poly}(m, n, \ell)$ bits.

You might need to prove and use the following fact: for a $k \times k$ matrix M with ℓ-bit integers, its inverse M^{-1} has rational entries with $\text{poly}(k, \ell)$ bits in both numberators and denominators.

Homework Problem 3. Prove Claim 1 in the light of above.