CS602 Applied Algorithms **2019-20 Sem II**

Lecture 16: March 12

Scribe: Aman Jain *Lecturer: Rohit Gurjar*

Steiner Tree/Forest: $(s_1, t_1), (s_1, t_1), ..., (s_k, t_k)$ are given. Interesting cuts : $S \subseteq V$ s.t. $\exists i$, $s_i \in S$ and $t_i \in \overline{S}$.

Primal Dual For Steiner Tree Problem

The Primal LP for our Steiner Tree problem as discussed in previous class will be:

$$\begin{array}{rcl} \min & \sum_{e \in E} w_e x_e & \text{s.t.} \\ \text{for each } e \in E, & x_e \geq 0 & \\ \text{for each interesting cut } S \subseteq V, & \sum_{e \in \delta(S)} x_e \geq 1 & \end{array}$$

Note that the constraint $x_e \leq 1$ would be redundant. Now, the corresponding Dual LP would be:

$$\begin{array}{rcl} \max & \sum_{\substack{S \subset V \\ S \text{ is interesting}}} y_S & \text{s.t.} \\ \text{for each interesting } S \subseteq V & y_S \geq 0, & \\ \text{and for each } e \in E, & \sum_{\substack{S \subseteq V \\ S \text{ is interesting} \\ \text{and } e \in \delta(S)}} y_S \leq w_e & \end{array}$$

Complementary Slackness conditions:

I Dual complementary slackness:

$$x_e \neq 0 \implies \sum_{\substack{S \subseteq V \\ S \text{ is interesting} \\ \text{and } e \in \delta(S)}} y_S = w_e$$

II Primal complementary slackness:

$$y_s \neq 0 \implies \sum_{e \in \delta(S)} x_e = 1$$

Primal Dual Algorithm

- Start with some feasible dual solution (say $y_s = 0 \forall s$). Look for primal feasible solution(integral) X which satisfies **I** (if an edge is tight ($\sum_S y_S = w_e$) you can take $x_e = 1$).
- If success, then done.
- If fail, then try improving the dual. How to do this? Suppose you find some S s.t. all edges in $\delta(S)$ are non-tight. Then increase y_s such that you get atleast one more tight edge. Now the question is how to find this S? Take the subgraph of tight edges G' and see if for each i, s_i is connected with t_i, $s_i \sim t_i$. If not, then there will be an interesting cut S such that $\delta(S)$ is empty in G', that is, all egdes in $\delta(S)$ are non-tight.

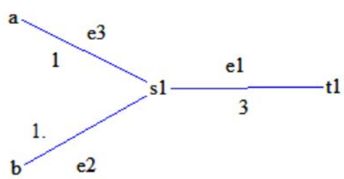
– Finally, say we have a primal feasible $\mathbf{x}$. Suppose x satisfies the primal complementary slackness approximately. That is, let γ be such that

$$y_s \neq 0 \implies \sum_{e \in \delta(S)} x_e \leq \gamma$$

Then,

- **Claim:** $\mathbf{x}$ is γ-optimal (Homework)

Example run of Algorithm



The following four will be interesting cuts with corresponding dual variables y_1, y_2, y_3, y_4.

y_1	$\{s_1\}$	$\{t_1, a, b\}$	$[e1, e2, e3]$	$x_{e_1} + x_{e_2} + x_{e_3} \geq 1$
y_2	$\{s_1, a\}$	$\{t_1, b\}$	$[e1, e2]$	$x_{e_1} + x_{e_2} \geq 1$
y_3	$\{s_1, b\}$	$\{t_1, a\}$	$[e1, e3]$	$x_{e_1} + x_{e_3} \geq 1$
y_4	$\{s_1, a, b\}$	$\{t_1\}$	$[e1]$	$x_{e_1} \geq 1$

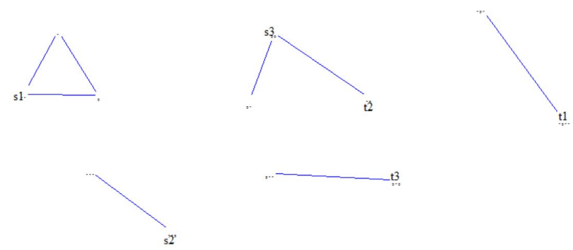
Dual :

- $y_1 + y_2 + y_3 + y_4 \leq 3$
- $y_1 + y_2 \leq 1$
- $y_1 + y_3 \leq 1$

Initially : $y_1 = y_2 = y_3 = y_4 = 0$. No tight edges.
Increase y_2 (chosen arbitrarily) : $y_1 = y_3 = y_4 = 0$, $y_2 = 1$. e_2 becomes tight. Set $x_{e_2} = 1$. Not feasible still. Improve the dual.
Increase y_4 : $y_1 = y_3 = 0$, $y_2 = 1$, $y_4 = 2$. e_1 becomes tight. Set $x_{e_1} = 1$. Feasible.
You can throw out e_2 out of the final solution.

Improvement ideas

1. Pruning: Once you get a feasible $\mathbf{x}$. go back in reverse order over all edges in $\mathbf{x}$ (reverse since the higher cost edges are added later). Set $x_e = 0$ if it remains feasible.
2. Increase many dual variables simultaneously.



Find current set of connected components (in the subgraph of tight edges). Consider interesting cuts $T_1, T_2, ..., T_m$
increasing all the dual y_{T_i} simultaneously.

More details will follow in the next lecture.