Steiner Tree/Forest: \((s_1, t_1), (s_1, t_1), \ldots, (s_k, t_k)\) are given. Interesting cuts : \(S \subseteq V\) s.t. \(\exists i, s_i \in S\) and \(t_i \in S\).

**Primal Dual For Steiner Tree Problem**

The Primal LP for our Steiner Tree problem as discussed in previous class will be:

\[
\min \sum_{e \in E} w_e x_e \\
\text{s.t.} \\
\text{for each } e \in E, \quad x_e \geq 0 \\
\text{for each interesting cut } S \subseteq V, \quad \sum_{e \in \delta(S)} x_e \geq 1
\]

Note that the constraint \(x_e \leq 1\) would be redundant. Now, the corresponding Dual LP would be:

\[
\max \sum_{S \subseteq V} y_S \\
\text{s.t.} \quad \text{for each interesting } S \subseteq V, \quad y_S \geq 0, \\
\text{and for each } e \in E, \quad \sum_{S \subseteq V} y_S \leq w_e \\
\text{and } e \in \delta(S)
\]

**Complementary Slackness conditions:**

I Dual complementary slackness:

\[x_e \neq 0 \implies \sum_{S \subseteq V} y_S = w_e \text{ with } S \text{ interesting and } e \in \delta(S)\]

II Primal complementary slackness:

\[y_s \neq 0 \implies \sum_{e \in \delta(S)} x_e = 1\]

**Primal Dual Algorithm**

- Start with some feasible dual solution (say \(y_s = 0\)s). Look for primal feasible solution(integral) \(X\) which satisfies I (if an edge is tight (\(\sum_S y_S = w_e\)) you can take \(x_e = 1\)).
- If success, then done.
- If fail, then try improving the dual. How to do this? Suppose you find some \(S\) s.t. all edges in \(\delta(S)\) are non-tight. Then increase \(y_s\) such that you get atleast one more tight edge. Now the question is how to find this \(S\)? Take the subgraph of tight edges \(G'\) and see if for each \(i, s_i\) connected with \(t_i, s_i \sim t_i\). If not, then there will be an interesting cut \(S\) such that \(\delta(S)\) is empty in \(G'\), that is, all egdes in \(\delta(S)\) are non-tight.
– Finally, say we have a primal feasible $x$. Suppose $x$ satisfies the primal complementary slackness approximately. That is, let $\gamma$ be such that

$$y_s \neq 0 \implies \sum_{e \in \delta(S)} x_e \leq \gamma$$

Then,

- **Claim:** $x$ is $\gamma$-optimal (Homework)

**Example run of Algorithm**

![Graph](image)

The following four will be interesting cuts with corresponding dual variables $y_1, y_2, y_3, y_4$.

<table>
<thead>
<tr>
<th>Cut</th>
<th>Dual Variables</th>
<th>Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$s_1$, $t_1, a, b$, $e_1, e_2, e_3$</td>
<td>$x_{e_1} + x_{e_2} + x_{e_3} \geq 1$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$s_1, a$, $t_1, b$, $e_1, e_2$</td>
<td>$x_{e_1} + x_{e_2} \geq 1$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$s_1, b$, $t_1, a$, $e_1, e_3$</td>
<td>$x_{e_1} + x_{e_3} \geq 1$</td>
</tr>
<tr>
<td>$y_4$</td>
<td>$s_1, a, b$, $t_1$, $e_1$</td>
<td>$x_{e_1} \geq 1$</td>
</tr>
</tbody>
</table>

**Dual:**

- $y_1 + y_2 + y_3 + y_4 \leq 3$
- $y_1 + y_2 \leq 1$
- $y_1 + y_3 \leq 1$

Initially: $y_1 = y_2 = y_3 = y_4 = 0$. No tight edges.
Increase $y_2$ (chosen arbitrarily): $y_1 = y_3 = y_4 = 0$, $y_2 = 1$. $e_2$ becomes tight. Set $x_{e_2} = 1$. Not feasible still.
Improve the dual.
Increase $y_4$ : $y_1 = y_3 = 0$, $y_2 = 1$, $y_4 = 2$. $e_1$ becomes tight. Set $x_{e_1} = 1$. Feasible.
You can throw out $e_2$ out of the final solution.

**Improvement ideas**

1. **Pruning:** Once you get a feasible $x$, go back in reverse order over all edges in $x$ (reverse since the higher cost edges are added later). Set $x_e = 0$ if it remains feasible.
2. Increase many dual variables simultaneously.
Find current set of connected components (in the subgraph of tight edges). Consider interesting cuts $T_1, T_2, \ldots, T_m$ increasing all the dual $y_{T_i}$ simultaneously.

More details will follow in the next lecture.