CS602 Applied Algorithms **2019-20 Sem II**

Lecture 2: January 20

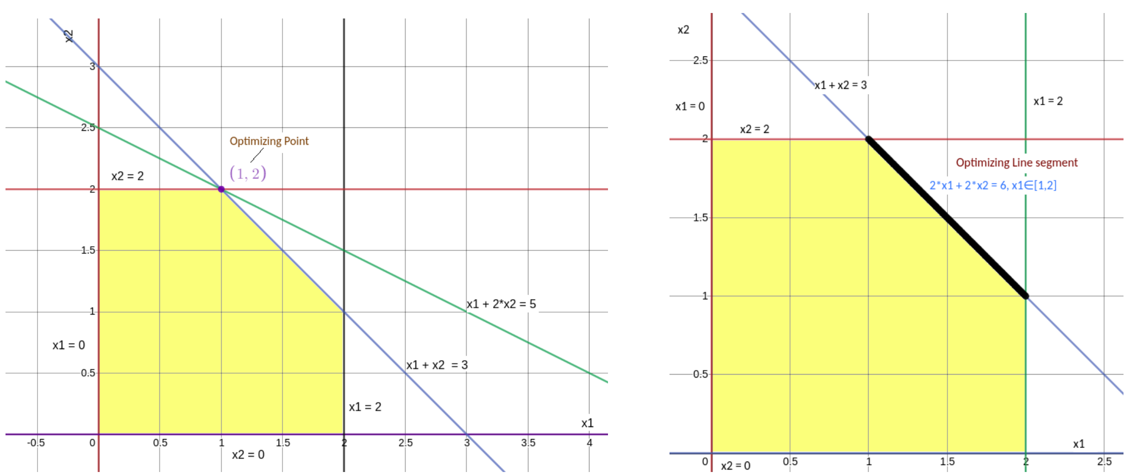
Scribe: Jay Bansal *Lecturer: Rohit Gurjar*

Maximizing set and faces

Definition 2.1 (Face of a Polyhedron)**.** *Let P be a polyhedron given by a set of linear constraints, i.e., $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, and let (A', b') be any subset of rows of (A, b). Then if $F = \{x \in P \mid A'x = b'\}$, and F is not empty, we call F a face of the polyhedron P. We say that the constraints $A'x \leq b'$ are tight for the face F (or every point in F).*

Example: Consider the following linear program. Let P be a polyhedron given by the following set of linear constraints (see Figure 1):

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_1 &\leq 2 \\ x_2 &\leq 2 \quad (1) \\ x_1 + x_2 &\leq 3 \quad (2) \end{aligned}$$



(a) A point maximizing function f_1 (b) A line segment maximizing function f_2

Figure 1: A linear function is optimized at a face

Now consider these two optimizing functions, with the above constraints:

$$\begin{aligned} f_1 &= x_1 + 2x_2 \\ f_2 &= 2x_1 + 2x_2 \end{aligned}$$

It can be seen from the Figure 1 that f_1 is maximized at a point in P which is a face of P (tight constraints are (1) and (2)) and f_2 is maximized at a line segment in P which is also a face of P (tight constraint is (1)). In general, maximizing set will be a face of any dimension (could be a corner, a line segment/edge of the polyhedron, or any $k(\leq n)$-dimensional face).

Definition 2.2 (Convex Set). *A set $P \subseteq \mathbb{R}^n$ is called convex if for all $a, b \in P$, any convex combination of a and b, that is, $a\lambda + (1-\lambda)b$ for $0 \leq \lambda \leq 1$ is also in P.*

Lemma 2.3. *The feasible region of a set of linear constraints, i.e., $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$, is a convex set.*

Proof. Let a_1 and a_2 be two points in P, that is, $Aa_1 \leq b$ and $Aa_2 \leq b$. Consider some $0 \leq \lambda \leq 1$.

$$A(\lambda a_1 + (1-\lambda)a_2) = \lambda A a_1 + (1-\lambda)Aa_2 \leq \lambda b + (1-\lambda)b = b$$

Thus, by the definition of a convex set, P is convex. □

Claim 2.4. *Suppose $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ and $z \in P$ is a point maximizing $w^\intercal x$. Let (A', b') be a subset of rows of (A, b) that gives all the tight constraints for z, i.e., $A'z = b'$. Then, any point $y \in P$ satisfying $A'y = b'$ will also maximize $w^\intercal x$ over P.*

Proof. Let $w^\intercal x = w^*$. Now, for the sake of contradiction, suppose $w^\intercal y < w^\intercal z = w^*$ This implies

$$w^\intercal(z - y) > 0 \tag{3}$$

Now, let us define a point $p = z + (z-y)\epsilon$ for some $\epsilon > 0$. Then, using Equation (3),

$$w^\intercal p = w^\intercal z + \epsilon w^\intercal(z-y) > w^\intercal z = w^* \tag{4}$$

We argue that $p \in P$. First, we show that the tight constraints for z are also tight for p.

$$A'p = A'z + A'[(z-y)\epsilon] = A'z = b'. \tag{5}$$

Now, consider any non-tight constraint for z, i.e., $a_i^\intercal z < b_i$. We have

$$a_i^\intercal p = a_i^\intercal z + a_i^\intercal[(z-y)\epsilon] < b_i \tag{6}$$

which holds because there always exists a small enough ϵ which makes it true. (5) and (6) prove our claim that $p \in P$. Combining it with (4), we get that the function $w^\intercal x$ takes a larger value at p than at z, which is contradiction. □

Claim 2.5. *If α and β are two points maximizing $w^\intercal x$ in P. Then $\frac{\alpha+\beta}{2}$ will also maximize $w^\intercal x$ in P.*

Proof. By Lemma 2.3, $\frac{\alpha+\beta}{2} \in P$. Let

$$w^\intercal \alpha = w^\intercal \beta = w^*.$$

Then

$$w^\intercal\left(\frac{\alpha+\beta}{2}\right) = w^\intercal\alpha/2 + w^\intercal\beta/2 = w^*$$

which makes $\frac{\alpha+\beta}{2}$ maximize $w^\intercal x$ in P. □

Claim 2.6. *For any $w \in \mathbb{R}^n$ and a polyhedron P, the set of points $x \in P$ maximizing $w^\intercal x$ forms a face.*

Proof. For any two optimal points which have different **smallest** face containing them, if both points optimize $w^\intercal x$, we need to prove that both of them lie on a face and that entire face optimizes $w^\intercal x$. If we prove this, we can take all possible such distinct points two at a time and recursively claim that the entire set of points in P maximizing $w^\intercal x$ forms a face of P.

Let $z_1 \in P$ and $z_2 \in P$ be two points that maximize $w^\intercal x$. By Claim 2.5, $(\frac{z_1+z_2}{2})$ is also optimal, and by Claim 2.4, all the points on the smallest face containing $(\frac{z_1+z_2}{2})$, say F are also optimal. Now, we just need to prove that z_1 and z_2 lie on F and we are done.

Let us say z_1 doesn't lie on F, then among the set of tight constraints for F, there must be one, say $a_i^\intercal x \leq b_i$, that is not tight for z_1. That is, $a_i^\intercal z_1 < b_i$. Since $(\frac{z_1+z_2}{2})$ lies on F and satisfies $a_i^\intercal(\frac{z_1+z_2}{2}) = b_i$, we get that $a_i^\intercal z_2 > b_i$. This is a contradiction, since $z_2 \in P$ and cannot violate $a_i^\intercal \leq b_i$. Thus z_1 and z_2 lie on F. □