CS602 Applied Algorithms **2019-20 Sem II**

Lecture 3: January 21

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Corners/Vertices of a Polyhedron

Lemma 3.1. *For any face F of P, $\exists w \in \mathbb{R}^n$ such that F is exactly the set of points maximizing $w^T x$ over P.*

Proof. Let polytope $P \subseteq \mathbb{R}^n$ be

$$\begin{aligned} a_1^T x &\leq b_1 \\ &\vdots \\ a_m^T x &\leq b_m \end{aligned}$$

and face F be the set of constraints

$$\begin{aligned} a_1^T x &= b_1 \\ a_2^T x &= b_2 \\ a_3^T x &\leq b_3 \\ &\vdots \\ a_m^T x &\leq b_m \end{aligned}$$

Then $w = a_1 + a_2$ is trivially the required w (As $a_1^T x + a_2^T x \leq b_1 + b_2 \ \forall x \in P$ and $a_1^T x + a_2^T x = b_1 + b_2$ only for $x \in F$). This can be extended for any face similarly. Hence, proved. $\square$

Corollary 3.2. *For $w^T x$, there is a corner of P which attains maximum value.*

Definition 3.3 (Corner/Vertex)**.** *For polytope $P \subseteq \mathbb{R}^n$*

$$\begin{aligned} a_1^T x &\leq b_1 \\ a_2^T x &\leq b_2 \\ &\vdots \\ a_m^T x &\leq b_m \end{aligned}$$

z is a vertex if $z \in P$ and if there is a subset of n linearly independent constraints which are tight for z.

Definition 3.4 (Corner/Vertex)**.** *z is a corner of P if $z \in P$ and $\forall y \in \mathbb{R}^n \setminus \{0\}$, $z + y \in P \Rightarrow z - y \notin P$.*

Definition 3.5 (Corner/Vertex)**.** *z is a corner of P if $z \in P$ and $\exists w \in \mathbb{R}^n$ such that z is the UNIQUE point maximizing $w^T x$ over P.*

Claim 3.6. *All 3 definitions of Corner/Vertex are equivalent*

Proof. $3.5 \Rightarrow 3.4$. $\exists w : w^T z = \alpha^*$
For contradiction, suppose $y \neq 0$ is such that $z + y \in P$ and $z - y \in P$.
$\Rightarrow w^T(z + y) \leq \alpha^*$, $w^T(z - y) \leq \alpha^*$
If one of the above is strictly less than α^* then other one would be greater, thus, they must be same as α^*.
Hence, we get a contradiction to the fact that z is the unique maximizing point. $\square$

$3.3 \Rightarrow 3.5$. Using *Lemma* 3.1, $\exists w \in \mathbb{R}^n$ such that z if exactly the set of points mazimizing $w^T x$, but by *Definition* 3.3, z is a UNIQUE point (n independent tight constraints in $\mathbb{R}^n$ correspond to a single point). Hence, this implies the condition in *Definition* 3.5. □

$3.4 \Rightarrow 3.3$. We want to show the condition in Definition 3.3. For the sake of contradiction, let us assume that the maximum number of constraints which are tight for z is k. Without loss of generality, say $a_i^T z = b_i$ for $1 \leq i \leq k$. If the rank of $(a_1, a_2, \ldots, a_k)$ is less than n then we know from linear algebra that there must be common orthogal vector to all of them. That is, there exists $\delta \in \mathbb{R}^n$ such that

$$a_i^T \delta = 0 \text{ for } 1 \leq i \leq k.$$

Now, choose two new points $z + \epsilon\delta$ and $z - \epsilon\delta$ for some small $\epsilon > 0$. The constraints which were tight for z will also be tight for these two points (from ()). Moreover, the constraints which were not tight for z can still be kept non-tight for the other two points by choosing a small enough ϵ. Thus, the two points are feasible. And, we get a contradiction to the condition that at least one of $z + y$ and $z - y$ should be outside the polyhedron. □

□