Corners/Vertices of a Polyhedron

Lemma 3.1. For any face $F$ of $P$, $\exists w \in \mathbb{R}^n$ such that $F$ is exactly the set of points maximizing $w^T x$ over $P$.

Proof. Let polytope $P \subseteq \mathbb{R}^n$ be

\[
\begin{align*}
    a_1^T x &\leq b_1 \\
    \vdots \\
    a_m^T x &\leq b_m
\end{align*}
\]

and face $F$ be the set of constraints

\[
\begin{align*}
    a_1^T x &= b_1 \\
    a_2^T x &= b_2 \\
    a_3^T x &\leq b_3 \\
    \vdots \\
    a_m^T x &\leq b_m
\end{align*}
\]

Then $w = a_1 + a_2$ is trivially the required $w$ (As $a_1^T x + a_2^T x \leq b_1 + b_2 \ \forall x \in P$ and $a_1^T x + a_2^T x = b_1 + b_2$ only for $x \in F$). This can be extended for any face similarly. Hence, proved.

Corollary 3.2. For $w^T x$, there is a corner of $P$ which attains maximum value.

Definition 3.3 (Corner/Vertex). For polytope $P \subseteq \mathbb{R}^n$

\[
\begin{align*}
    a_1^T x &\leq b_1 \\
    a_2^T x &\leq b_2 \\
    \vdots \\
    a_m^T x &\leq b_m
\end{align*}
\]

$z$ is a vertex if $z \in P$ and if there is a subset of $n$ linearly independent constraints which are tight for $z$.

Definition 3.4 (Corner/Vertex). $z$ is a corner of $P$ if $z \in P$ and $\forall y \in \mathbb{R}^n \setminus \{0\}$, $z + y \in P \Rightarrow z - y \notin P$.

Definition 3.5 (Corner/Vertex). $z$ is a corner of $P$ if $z \in P$ and $\exists w \in \mathbb{R}^n$ such that $z$ is the UNIQUE point maximizing $w^T x$ over $P$.

Claim 3.6. All 3 definitions of Corner/Vertex are equivalent

Proof. 3.5 $\Rightarrow$ 3.4. $\exists w : w^T z = \alpha^*$

For contradiction, suppose $y \neq 0$ is such that $z + y \in P$ and $z - y \in P$.

$\Rightarrow w^T(z + y) \leq \alpha^* \ , \ w^T(z - y) \leq \alpha^*$

If one of the above is strictly less than $\alpha^*$ then other one would be greater, thus, they must be same as $\alpha^*$.

Hence, we get a contradiction to the fact that $z$ is the unique maximizing point.
3.3 ⇒ 3.5. Using Lemma 3.1, \( \exists w \in \mathbb{R}^n \) such that \( z \) if exactly the set of points maximizing \( w^T x \), but by Definition 3.3, \( z \) is a UNIQUE point (\( n \) independent tight constraints in \( \mathbb{R}^n \) correspond to a single point). Hence, this implies the condition in Definition 3.5.

3.4 ⇒ 3.3. We want to show the condition in Definition 3.3. For the sake of contradiction, let us assume that the maximum number of constraints which are tight for \( z \) is \( k \). Without loss of generality, say \( a_i^T z = b_i \) for \( 1 \leq i \leq k \). If the rank of \( (a_1, a_2, \ldots, a_k) \) is less than \( n \) then we know from linear algebra that there must be common orthogonal vector to all of them. That is, there exists \( \delta \in \mathbb{R}^n \) such that

\[
a_i^T \delta = 0 \quad \text{for} \quad 1 \leq i \leq k.
\]

Now, choose two new points \( z + \epsilon \delta \) and \( z - \epsilon \delta \) for some small \( \epsilon > 0 \). The constraints which were tight for \( z \) will also be tight for these two points (from ()). Moreover, the constraints which were not tight for \( z \) can still be kept non-tight for the other two points by choosing a small enough \( \epsilon \). Thus, the two points are feasible. And, we get a contradiction to the condition that at least one of \( z + y \) and \( z - y \) should be outside the polyhedron.