CS602 Applied Algorithms **2019-20 Sem II**

Lecture 4: January 23

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Definition 4.1 (Convex Hull). *For a set of points $S = \{P_1, P_2, ..., P_k\}$, $S \subseteq \mathbb{R}^n$, Convex Hull is defined as the set of all points generated by convex combinations of these points, i.e.,*
$ConvexHull\{S\} = \{\lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_k P_k \mid \lambda_1, \lambda_2, ..., \lambda_k \geq 0$ *and* $\lambda_1 + \lambda_2 + ... + \lambda_k = 1\}$.

Example: Convex Hull of two points consists of all the points on the line joining the two points. Similarly, convex hull of n points is a polytope having all or some of those n points as vertices.

Claim 4.2. *A polytope is a Convex Hull of its (finitely many) vertices.*

Note: Since every vertex comes from a subset of the linear constraints $Ax \leq b$ being tight and there are only finitely many such subsets (not all give a vertex), we have finitely many vertices.

Theorem 4.3 (Carthéodory Theorem). *For a polytope $P \subseteq \mathbb{R}^n$, any point $z \in P$ can be written as a convex combination of at most $n+1$ vertices of P.*

Proof. Proof by induction:
Base case: On triangulating a polytope in $\mathbb{R}^2$, any point inside a polytope will be enclosed in a triangle formed by three of its vertices and hence can be written as their convex combination.
Inductive step: Let the theorem hold for $\mathbb{R}^{n-1}$. Take any arbitrary vertex v of P. Join v with z by a line and extend it to the other side till it hits a face F. Let the point on F where this line meets be y. F is at most an $n-1$ dimensional polytope.
We can write, $z = \lambda v + (1-\lambda)y$, $\lambda \geq 0$ where $y = \sum_{i=1}^{n} \lambda_i v_i$ such that $\forall i$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and v_i is a vertex of P.
Since y is a convex combination of at most n vertices of F, z becomes a convex combination of at most $n+1$ vertices as $\lambda + (1-\lambda)\sum_{i=1}^{n} \lambda_i = 1$ and $\forall i$, $(1-\lambda)\lambda_i \geq 0$. □

Claim 4.4. *If P is a Convex Hull of some finitely many points, then P can be described by some finitely many linear constraints.*

Proof. Let the given set have k points $\{(p_{i,1}, p_{i,2}, \ldots, p_{i,n}) \mid 1 \leq i \leq k\}$. Then their convex hull P can be described as follows: set of all points $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that there exists

$$0 \leq \lambda_1, \lambda_2, \ldots, \lambda_k \text{ such that}$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1 \text{ and}$$

$$x_j = \lambda_1 p_{1,j} + \lambda_2 p_{2,j} + \cdots + \lambda_k p_{k,j} \text{ for each } 1 \leq j \leq n.$$

This already gives us a description of P with linear constraints. However, the description uses extra variables $\lambda_1, \lambda_2, \ldots, \lambda_k$ besides the x_j variables. We want a description using just the n variables $(x_1, x_2, \ldots, x_n)$, since it is a polytope in $\mathbb{R}^n$.

We can eliminate the λ_i variables using Fourier Motzkin Elimination which aims to remove one variable at a time from a set of equations. In general, if we have the below equations and we want to remove λ,

$$\lambda \geq E1, \lambda \geq E2, \lambda \leq E3, \lambda \leq E4, \lambda = E5, \lambda = E6$$

We take all combinations of the RHS expressions and replace the above equations with:

$$E5 = E6, E5 \geq E1, E5 \geq E2, E5 \leq E3, E5 \leq E4$$

Note: The RHS expressions should not contain λ. Using this repeatedly for all λ_i's will give us the desired description. □

Example: We have two points $v_1 = (2, 1)$ and $v_2 = (1, 2)$ and we want to find the linear constraints that describe their Convex Hull P.

$$P = \{\lambda_1 v_1 + \lambda_2 v_2 | \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1\}$$

If (x_1, x_2) represents a point in P, then:

$$x_1 = 2\lambda_1 + \lambda_2$$
$$x_2 = \lambda_1 + 2\lambda_2$$

Now to remove λ_1, we rewrite the equations as the following:

$$\lambda_1 = (1/2)x_1 - (1/2)\lambda_2 \tag{1}$$
$$\lambda_1 = x_2 - 2\lambda_2 \tag{2}$$
$$\lambda_1 \geq 0 \tag{3}$$
$$\lambda_1 = 1 - \lambda_2 \tag{4}$$
$$\lambda_2 \geq 0 \tag{5}$$

Combining RHS of (4) with (2), (2) with (1) and (3) with (4), we get rid of equations (1), (2), (3), (4) to get:

$$\lambda_2 = x_2 - 1 \tag{6}$$
$$\lambda_2 = (2/3)x_2 - (1/3)x_1 \tag{7}$$
$$\lambda_2 \leq 1 \tag{8}$$

To get rid of λ_2, we combine RHS of (6) with (5), (6) with (8) and (6) with (7) to get:

$$x_1 + x_2 = 3 \tag{9}$$
$$x_2 \geq 1 \tag{10}$$
$$x_2 \leq 2 \tag{11}$$

Our final answer consists of (9), (10) and (11).