Feasibility $\iff$ Optimization
Optimization reduces to feasibility via binary search.

**Feasibility**
\[
\begin{align*}
A'x &\leq b' \\
W^T x &\geq W_o \\
&\quad \text{if feasible then pick } W_1 > W_o \\
&\quad \text{if not feasible then pick } W_1 < W_o
\end{align*}
\]

**Optimization**
\[
A'x \leq b' \quad \text{max } W^T x
\]

Terminate when
\[
\begin{align*}
&\quad W^T x \geq W_o \implies \text{feasible} \\
&\quad W^T x \leq W_o + \epsilon \implies \text{not feasible}
\end{align*}
\]

where $\epsilon \to \frac{1}{\exp(m,n)}$

**HW** Prove that number of steps after which algorithm stops is equal to $\text{poly}(m,n)$.

**Feasibility algorithm**

**Given Input** Matrix $A$ of size $m \times n$, with each entry being at most $B$.

**Algorithm** We can look for the vertices of the polyhedron $Ax \leq b$

To look for vertices, choose a subset of size $n$ say $A', b'$. Then solve $A'x = b'$ and check whether the solution satisfies other inequalities i.e., $Ax \leq b$

Since we are searching among all the subsets, time complexity of the above algorithm will be $\binom{m}{n}$ which is exponential.

If no such vertices exists then number of linearly independent constraints $< \text{number of variables}$. In this case, we can reduce the number of variables in the LP. Replace each LHS expression with a new variable.

\[
\begin{align*}
y_1 &= a^T_1 x \leq b_1 \\
y_2 &= a^T_2 x \leq b_2 \\
y_3 &= a^T_3 x \leq b_3 \\
&\quad \cdots
\end{align*}
\]

But if $a^T_3$ is dependent on $a^T_1$ and $a^T_2$ then $a^T_3$ can be written as linear combination of $a^T_1$ and $a^T_2$ which implies that $y_3$ can be written as a linear combination of $y_1$ and $y_2$, thus in this way we can reduce number of variables.
Feasibility ∈ NP?

If $Ax \leq b$ is feasible then can someone give easily verifiable proof/certificate for this?

The certificate for feasibility is a solution $x$ which satisfies $Ax \leq b$

But then how do we ensure that $x$ is polynomial many bits?

- If $x$ is a vertex then there are $n$ linearly independent tight constraints for $x$.
  $$A'x = b \implies x = A'^{-1}b$$

  and therefore number of bits in $x$ is polynomial of input size.

- If $x$ is not a vertex then we can make use of the following result.

  **Claim:** If there is a solution of $Ax \leq b$ then there exists a subset $(A', b')$ s.t. any solution of $A'x = b'$ is a feasible solution. (Proved in the lecture notes.)

  The solution to the equality $A'x = b'$ can now be used as the certificate of feasibility. The bit complexity can again be bounded using the fact that inverse of a matrix has polynomially bounded bit complexity.

This shows that feasibility ∈ NP.

Feasibility ∈ coNP?

If $Ax \leq b$ is not feasible then can someone give easily verifiable proof/certificate for this?

Ex: The following system of inequalities is not feasible.

$$x_1 + x_2 \geq 1 \quad (1)$$
$$x_1 \leq 0 \quad (2)$$
$$x_2 \leq 0 \quad (3)$$

The proof of the same is given below

Add eqn (2) and (3)

$$x_1 + x_2 \leq 0 \quad (4)$$

From eqn (1) and (4)

$$-1 \geq 0 \rightarrow contradiction \quad (5)$$

But can we find the proof in general form of any given system of inequalities? (To be continued in next lecture.)