CS602 Applied Algorithms **2019-20 Sem II**

Lecture 5: January 27

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Feasibility $\iff$ Optimization
Optimization reduces to feasibility via binary search.

Feasibility

$A'x \leq b'$
$W^T x \geq W_o$

- if feasible then pick $W_1 > W_o$
- if not feasible then pick $W_1 < W_o$

Terminate when

- $W^T x \geq W_o \implies$ feasible
- $W^T x \leq W_o + \epsilon \implies$ not feasible

where $\epsilon \to \frac{1}{exp(m,n)}$

Optimization

$A'x \leq b'$
$\max W^T x$

HW Prove that number of steps after which algorithm stops is equal to poly(m, n).

Feasibility algorithm

Given Input Matrix A of size $m \times n$, with each entry being at most B.

Algorithm We can look for the vertices of the polyhedron $Ax \leq b$

To look for vertices, choose a subset of size n say A', b'. Then solve $A'x = b'$ and check whether the solution satisfies other inequalities i.e., $Ax \leq b$

Since we are searching among all the subsets, time complexity of the above algorithm will be $\binom{m}{n}$ which is exponential.

If no such vertices exists then number of linearly independent constraints < number of variables. In this case, we can reduce the number of variables in the LP. Replace each LHS expression with a new variable.

$$y_1 = a_1^T x \leq b_1$$
$$y_2 = a_2^T x \leq b_2$$
$$y_3 = a_3^T x \leq b_3$$
$$\vdots$$

But if a_3^T is dependent on a_1^T and a_2^T then a_3^T can be written as linear combination of a_1^T and a_2^T which implies that y_3 can be written as a linear combination of y_1 and y_2, thus in this way we can reduce number of variables.

Feasibility $\in$ NP?

If $Ax \leq b$ is feasible then can someone give easily verifiable proof/certificate for this?

The certificate for feasibility is a solution x which satisfies $Ax \leq b$
But then how do we ensure that x is polynomial many bits?

- If x is a vertex then there are n linearly independent tight constraints for x.

$$A'x = b \Longrightarrow x = A'^{-1}b$$

  and therefore number of bits in x is polynomial of input size.

- If x is not a vertex then we can make use of the following result.

  Claim: If there is a solution of $Ax \leq b$ then there exists a subset (A', b') s.t. any solution of $A'x = b'$ is a feasible solution. (Proved in the lecture notes.)

  The solution to the equality $A'x = b'$ can now be used as the certificate of feasibility. The bit complexity can again be bounded using the fact that inverse of a matrix has polynomially bounded bit complexity.

This shows that feasibility $\in$ NP.

Feasibility $\in$ coNP?

If $Ax \leq b$ is not feasible then can someone give easily verifiable proof/certificate for this?

Ex: The following system of inequalities is not feasible.

$$x_1 + x_2 \geq 1 \quad (1)$$
$$x_1 \leq 0 \quad (2)$$
$$x_2 \leq 0 \quad (3)$$

The proof of the same is given below

Add eqn (2) and (3)

$$x_1 + x_2 \leq 0 \quad (4)$$

From eqn (1) and (4)

$$-1 \geq 0 \rightarrow \textit{contradiction} \quad (5)$$

But can we find the proof in general form of any given system of inequalities? (To be continued in next lecture.)