CS602 Applied Algorithms **2019-20 Sem II**

Lecture 6: January 28

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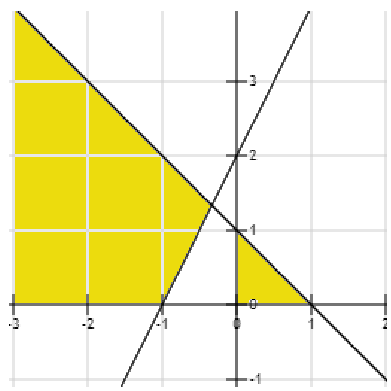
Binary Search Puzzle: You need to find an unknown rational number p/q (assume $p, q \in Z$, $p, q \leq M$). You are allowed queries like 'is $\alpha \geq p/q$'?

Feasibility in coNP

Example: Consider the following linear program. Let P be a polyhedron given by the following set of linear constraints

$$\begin{aligned} x_1 &\geq 0 & (1)\\ x_2 &\geq 0 & (2)\\ x_1 + x_2 &\leq 1 & (3)\\ x_2 - 2x_1 &\geq 2 & (4) \end{aligned}$$

As from the below figure we can state that there is no feasible solution for this LP.



We can also prove this Algebraically, (1) * 3 + (4) will give $x_1 + x_2 \geq 2$.

$$\begin{aligned} x_1 + x_2 &\geq 2 & (5)\\ -x_1 - x_2 &\geq -1 & (6) \end{aligned}$$

Now adding both (5) and (6) will give $0 \geq 1$ which is incorrect. Therefore the given LP is not feasible. We will now prove this for any linear program.

Definition 6.1 (Cone). *$C \in \mathbb{R}^n$ is a cone if for every $z \in C$, $\alpha z \in C$, $\forall \alpha \geq 0$*

Definition 6.2 (Convex cone). *A cone $C \in \mathbb{R}^n$ is a convex cone if $\alpha x + \beta y$ belongs to C, for any positive scalars α, β, and any x, y in C.*

Definition 6.3 (Polyhedral Cone). *A cone C is called Polyhedral Cone if it can be represented by a finite set of vectors $v_1, v_2 \cdots v_k$ such that C is the conical combination of $\{v_1, v_2 \cdots v_k\}$, i.e., $C = \{a_1 v_1 + \cdots + a_k v_k \mid a_i \geq 0, v_i \in \mathbb{R}^n\}$*

Lemma 6.4. *A Polyhedral Cone can be described by $Ax \leq \mathbf{0}$ $\{A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n\}$.*

A proof of this lemma can be obtained by Fourier-Motzkin elimination, just like we did in case of a convex hull of points (Lecture 4).

Corollary 6.5. *For a polyhedral cone $C = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$ where $v_i \in \mathbb{R}^n, \alpha_i \geq 0$, If $u \notin C$ for some $u \in \mathbb{R}^n$ then $\exists a \in \mathbb{R}^n$ such that $a^\mathsf{T} u > 0$ but $a^\mathsf{T} z \leq 0, \forall z \in C$.*

Lemma 6.6 (Farkas' Lemma)**.** *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If $Ax \leq b$ is not feasible then $\exists y \in \mathbb{R}^m$, $y = (y_1, \cdots, y_m) \geq 0$ such that $y^\mathsf{T} A = \boldsymbol{O}$ and $y^\mathsf{T} b = -1$.*

Farkas' Lemma from Corollary

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. As $Ax \leq b$ is composed of m constraints, it can be written individually as

$$\begin{aligned} a_1^\mathsf{T} x - b_1 &\leq 0 \\ a_2^\mathsf{T} x - b_2 &\leq 0 \\ &\vdots \\ a_m^\mathsf{T} x - b_m &\leq 0 \end{aligned}$$

where $a_i \in \mathbb{R}^n$. Now define a vector $v_i = (a_i, b_i)$ i.e., $\{a_{i1}, a_{i2}, \cdots, a_{in}, b_i\}$.
Defining C as conical combination of vectors v_i, i.e., $C = cone(v_1, v_2, \cdots, v_m)$. If $Ax \leq b$ is not feasible then we need to to show that the vector $\{0, 0 \cdots, 0, -1\}$ belongs to the set C. We will use Corollary 6.5 for this.

Suppose $u = \{0, 0 \cdots, 0, -1\}$, for contradiction suppose $u \notin C$ then by corollary 6.5 $\exists \beta \in \mathbb{R}^{n+1}$ such that:

$$\beta^\mathsf{T} u > 0 \tag{7}$$

$$\beta^\mathsf{T} v_i \leq 0 \text{ for each } 1 \leq i \leq m. \tag{8}$$

Write $\beta = \{\lambda, \beta_{n+1}\}$. By (7) and (8) we get:

$$\beta_{n+1} < 0 \tag{9}$$

$$\lambda^\mathsf{T} a_i + \beta_{n+1} b_i \leq 0 \tag{10}$$

Divide equation (10) by $-\beta_{n+1}$ and let $\gamma = -\lambda / \beta_{n+1}$. Now,

$$\gamma^\mathsf{T} a_i - b_i \leq 0 \tag{11}$$

$$\gamma^\mathsf{T} a_i \leq b_i \tag{12}$$

Equation (12) can be written as

$$a_i^\mathsf{T} \gamma \leq b_i \tag{13}$$

which shows that γ is a feasible solution which is a contradiction to the assumption in the Lemma 6.6. Therefore $u \in C$ which proves the Farkas' lemma.

Farkas' Lemma Another Form

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. If the system $x \in \mathbb{R}^n$, $Ax = b, x \geq 0$ is not feasible then $\exists y = \{y_1, y_2, \cdots, y_m\}$ where $y_i \in \mathbb{R}$ such that

$$y^\mathsf{T} A \geq 0 \tag{14}$$

$$y^\mathsf{T} b < 0 \tag{15}$$

Separating hyperplane

Definition 6.7 (Separating Hyperplane)**.** *Let $C \in \mathbb{R}^n$ be a closed convex set and $x \in \mathbb{R}^n$ not in C. Then x and C can be strictly separated by a separating hyperplane.*

Proof of Separating Hyperplane

Let $C \in \mathbb{R}^n$ be a closed convex set and $p \in \mathbb{R}^n$ not in C. Let $z \in C$ be the closest point in C from p. The equation of the line joining the points p and z is parallel to $z - p$. Now,the hyperplane perpendicular to the line joining p and z and passing through z has the equation :

$$(z-p)^{\mathsf{T}}x \quad = \quad (z-p)^{\mathsf{T}}z \tag{16}$$

Here x is any point on that hyperplane with equation (16).

To prove: For any point $x \in C$, $(z-p)^{\mathsf{T}}x \geq (z-p)^{\mathsf{T}}z$.
For contradiction suppose not, then $\exists x$ such that

$$(z-p)^{\mathsf{T}}x \quad < \quad (z-p)^{\mathsf{T}}z \tag{17}$$

As C is convex, now for every $\lambda \in [0,1]$, point $\lambda x + (1-\lambda)z$ lies in C. Now the distance between this point and p will be:

$$= ||p - (\lambda x + (1-\lambda)z)||^2 \tag{18}$$

$$= ||p-z||^2 + \lambda^2||x-z||^2 - 2\lambda(z-p)^{\mathsf{T}}(z-x) \tag{19}$$

Note that the term $2\lambda(z-p)^{\mathsf{T}}(z-x)$ is positive. We can choose our λ to be small enough so that the term with λ^2 will be negligible and thus, the distance between $\lambda x + (1-\lambda)z$ and p will be less than z and p i.e., $||p-z||^2$. This is a contradiction as z has the smallest distance in C from p. Therefore, $\forall x \in C$

$$(z-p)^{\mathsf{T}}x \quad \geq \quad (z-p)^{\mathsf{T}}z \tag{20}$$

Assuming $z - p = a$ and $(z-p)^{\mathsf{T}}z = b$,we have equation

$$a^{\mathsf{T}}x \quad - \quad b \tag{21}$$

So, $\forall x \in C, a^{\mathsf{T}}x - b \geq 0$ from equation 20.
For $x = p$, $a^{\mathsf{T}}x - b < 0$ as $-(z-p)^{\mathsf{T}}(z-p) < 0$ .Therefore $a^{\mathsf{T}}x - b$ will act as a hyperplane separating point p from all points $x \in C$.