**CS602 Applied Algorithms** — **2019-20 Sem II**

# Lecture 7: January 30

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Feasibility $\in$ co-NP and Feasibility $\in$ NP. And if a problem is in NP and also co-NP, it mostly turns out to be in P.

## Optimization(Duality)

**Example:** Consider the following linear program. Let $P$ be a polyhedron given by the following set of linear constraints

$$
\begin{aligned}
x &\geq 0 && (1)\\
y &\geq 0 && (2)\\
x &\leq 2 && (3)\\
y &\leq 2 && (4)\\
x+y &\leq 3 && (5)
\end{aligned}
$$

Suppose we have to maximise $f(x,y) = 7x + 8y$ under these constraints. Let say the optimal value is $w^*$. Take any point in $P$. Let's pick $(1,1) \in P$, we can confirm that $f(1,1) = 15$. And hence, $w^* \geq 15$. Similarly, any feasible point of the LP will us a lower bound on the optimal value.

**Observation 7.1** *In the LP* $\max\{w^\intercal x \mid Ax \leq b\}$ *if $\alpha$ is a feasible point then* $w^* \geq w^\intercal \alpha$.

For example, $(2,1)$ is a feasible point and it gives us $w^* \geq f(2,1) = 22$. Another point $(1,2)$ gives a better lower bound: $w^* \geq f(1,2) = 23$.

Can we also get an upper bound on $w^*$ somehow? To upper bound $w^*$ we can try to express $f(x,y)$ as positive linear combination of given linear constraints. Let us do it for the above example.

Attempt 1: Let us multiply (3) by 7 and (4) by 8, and add the two inequalites.

$$
\begin{aligned}
7\times \quad (x &\leq 2)\\
8\times \quad (y &\leq 2)\\
\hline
7x+8y &\leq 14+16\\
\implies f(x,y) &\leq 30
\end{aligned}
$$

This gives us an upper bound of $w^* \leq 30$ which is far from the lower bounds we saw. Let us try another possibility.

Attempt 2: Let us multiply (5) by 8 and (1) by -1, and add the two.

$$
\begin{aligned}
8\times \quad (x+y &\leq 3)\\
-1\times \quad (x &\geq 0)\\
\hline
7x+8y &\leq 24-0\\
\implies f(x,y) &\leq 24
\end{aligned}
$$

We get $w^* \leq 24$ which is close to the lower bound of 23, but there is still a gap.

Attempt 3: Let us multiply (5) by 7 and (3) by 1, and add the two.

$$
\begin{aligned}
7\times \quad (x+y &\leq 3)\\
1\times \quad (x &\leq 2)\\
\hline
\end{aligned}
$$

$$
\begin{aligned}
7x + 8y &\leq 21 + 2 \\
\implies f(x, y) &\leq 23
\end{aligned}
$$

We get $w^* \leq 23$, which matches exactly with our best lower bound. Of course, we can not hope to further improve either the upper bound or the lower bound. Thus, it must be that $w* = 23$ and $(1, 2)$ must be the optimal point.

# Weak Duality Theorem

This way of upper bounding the optimal value can be expressed as another LP, which is called the dual LP.

**Primal LP (LP)** : maximise $f(x) = w^{\mathsf{T}}x$ such that $Ax \leq b$ where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^m$, $w \in \mathbb{R}^n$.

**Dual LP (LP*)** : minimise $g(y) = b^{\mathsf{T}}y$ such that $A^{\mathsf{T}}y = w$, $y \geq 0$.

**Theorem:** if $(x_1, x_2, ...., x_n)$ is a feasible solution for the primal maximization linear program and $(y_1, y_2, ...., y_m)$ is a feasible solution for the dual minimization linear program, then the weak duality theorem can be stated as $\sum_{j=1}^{n} w_j x_j \leq \sum_{i=1}^{m} b_i y_i$.

**Proof:** Suppose $\alpha$ is feasible for LP and $\beta$ is feasible for LP*

$$
\begin{aligned}
A\alpha &\leq b \\
\beta^{\mathsf{T}}A\alpha &\leq \beta^{\mathsf{T}}b && (\textit{because } \beta \geq 0) \\
w^{\mathsf{T}}\alpha &\leq \beta^{\mathsf{T}}b && (A^{\mathsf{T}}\beta = w) \\
\therefore f(\alpha) &\leq g(\beta)
\end{aligned}
$$

# Strong Duality Theorem

If the LP is feasible and optimal value $w^*$ is bounded then $\exists \beta^* \in \mathbb{R}^m$ such that $A^{\mathsf{T}}\beta^* = w$ and $\beta^* \geq 0$ with $b^{\mathsf{T}}\beta^* = w^*$, i.e., OPT(LP) = OPT(LP*).