1 Weak Duality

Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \) and \( w \in \mathbb{R}^n \).

**Primal LP** - \( \max w^T x : Ax \leq b, x \in \mathbb{R}^n \)

**Dual LP** - \( \min b^T y : A^T y = w, y \geq 0, y \in \mathbb{R}^m \)

**Weak Duality:** \( \text{OPT(Primal LP)} \leq \text{OPT(Dual LP)}. \)

**Proof:** Let \( \alpha \) be a feasible solution for the primal LP whereas \( \beta \) be a feasible solution for the dual LP.

\[
A\alpha \leq b \\
\beta^T A\alpha \leq \beta^T b \quad (\because \beta \geq 0) \\
w^T \alpha \leq \beta^T b \quad (\because A^T \beta = w)
\]

The above inequality, in particular, is also true when \( \alpha, \beta \) are optimal solutions.

2 Strong Duality

Strong duality says that for optimal solutions the above inequality is, in fact, an equality. If LP is feasible and has a bounded optimal value, then there exists a dual feasible solution that has the same value as the primal optimal value.

**Claim 8.1.** Let \( \alpha^* \) be a optimal solution and \( w^* = w^T \alpha^* \) be the optimal value for the primal LP. Then

\[ \exists \beta^* \in \mathbb{R}^m \text{ s.t. } A^T \beta^* = w, \beta^* \geq 0 \text{ (dual feasible), and } b^T \beta^* = w^*. \]

**Proof.** Recall the corollary from previous class, that gives a separating hyperplane for a cone. For any cone \( C = \text{cone}(v_1, v_2, ..., v_k) \in \mathbb{R}^n \) and for a vector \( u \in \mathbb{R}^n \),

\[ u \notin C \implies \exists q \in \mathbb{R}^n \text{ s.t. } q^T u > 0 \text{ and } q^T v_i \leq 0, \forall 1 \leq i \leq k. \]

Consider the augmented matrix \((A|b)\). We want to show that there is a \( \beta^* \geq 0 \) such that

\[ (A|b)^T \beta^* = (w, w^*). \]

Here \((w, w^*) \in \mathbb{R}^{n+1}\) denotes a column vector whose first \(n\) entries come from \(w\) and the last entry is \(w^*\).

Let \((a_1, b_1), (a_2, b_2), ..., (a_m, b_m) \in \mathbb{R}^{n+1}\) be the columns of \((A|b)^T\). What is want to show is equivalent to

\[ (w, w^*) \in \text{cone}((a_1, b_1), (a_2, b_2), ..., (a_m, b_m)). \]

For the sake of contradiction, let it not be true. Then by above mentioned corollary, there exist \( q \in \mathbb{R}^n \) and \( r \in \mathbb{R} \) such that

\[ (q, r)^T(w, w^*) > 0 \text{ and } (q, r)^T(a_i, b_i) \leq 0, \forall 1 \leq i \leq m. \]

Equivalently,

\[ q^T w > -rw^* \text{ and } q^T a_i \leq -rb_i, \forall 1 \leq i \leq m. \]

Which is same as

\[ w^T q > -rw^* \text{ and } a_i^T q \leq -rb_i, \forall 1 \leq i \leq m. \]  \( (1) \)

Now, we consider two cases depending on the sign of \( r \). In both the cases, we will show that there is a primal feasible solution \( \alpha \) with value greater than \( w^* \), contradicting the optimality of \( \alpha^* \).
Case 1: $r < 0$. Divide (1) by $-r$ and define $\alpha = q/r$. We get

$$a_i^T \alpha \leq b_i \forall i, \text{ and } w^T \alpha > w^\star,$$

which is a contradiction.

Case 2: $r \geq 0$, which means $r + 1 > 0$. Now, consider $\alpha = (r + 1)\alpha^\star + q$. Since $\alpha^\star$ is primal optimal, we know

$$(r + 1)a_i^T \alpha^\star \leq (r + 1)b_i, \forall i, \text{ and } (r + 1)w^T \alpha^\star = (r + 1)w^\star.$$ Combining this with (1), we get

$$a_i^T \alpha = (r + 1)a_i^T \alpha^\star + a_i^T q \leq (r + 1)b_i - rb_i = b_i, \forall i.$$ And

$$w^T \alpha = (r + 1)w^T \alpha^\star + w^T q > (r + 1)w^\star - rw^\star = w^\star.$$ This gives us a contradiction.

3 Various Forms of LP

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<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
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<tr>
<td>$\max w^T x : Ax \leq b$</td>
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