CS602 Applied Algorithms **2019-20 Sem II**

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1 Weak Duality

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$.

Primal LP - $\max\ w^\intercal x : Ax \leq b,\ x \in \mathbb{R}^n$

Dual LP - $\min\ b^\intercal y : A^\intercal y = w,\ y \geq 0,\ y \in \mathbb{R}^m$

Weak Duality: OPT(Primal LP) $\leq$ OPT(Dual LP).

Proof: Let α be a feasible solution for the primal LP whereas β be a feasible solution for the dual LP.

$$\begin{aligned} A\alpha &\leq b \\ \beta^\intercal A\alpha &\leq \beta^\intercal b \quad (\because \beta \geq 0) \\ w^\intercal \alpha &\leq \beta^\intercal b \quad (\because A^\intercal \beta = w) \end{aligned}$$

The above inequality, in particular, is also true when α, β are optimal solutions.

2 Strong Duality

Strong duality says that for optimal solutions the above inequality is, in fact, an equality. If LP is feasible and has a bounded optimal value, then there exists a dual feasible solution that has the same value as the primal optimal value.

Claim 8.1. *Let α^* be a optimal solution and $w^* = w^\intercal \alpha^*$ be the optimal value for the primal LP. Then*

$$\exists \beta^* \in \mathbb{R}^m \text{ s.t. } A^\intercal \beta^* = w,\ \beta^* \geq 0 \text{ (dual feasible)},\ \text{and } b^\intercal \beta^* = w^*.$$

Proof. Recall the corollary from previous class, that gives a separating hyperplane for a cone. For any cone $C = \text{cone}(v_1, v_2, ..., v_k) \in \mathbb{R}^n$ and for a vector $u \in \mathbb{R}^n$,

$$u \notin C \implies \exists q \in \mathbb{R}^n \text{ s.t. } q^\intercal u > 0 \text{ and } q^\intercal v_i \leq 0,\ \forall 1 \leq i \leq k.$$

Consider the augmented matrix $(A|b)$. We want to show that there is a $\beta^* \geq 0$ such that

$$(A|b)^\intercal \beta^* = (w, w^*).$$

Here $(w, w^*) \in \mathbb{R}^{n+1}$ denotes a column vector whose first n entries come from w and the last entry is w^*. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m) \in \mathbb{R}^{n+1}$ be the columns of $(A|b)^\intercal$. What is want to show is equivalent to

$$(w, w^*) \in \text{cone}((a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)).$$

For the sake of contradiction, let it not be true. Then by above mentioned corollary, there exist $q \in \mathbb{R}^n$ and $r \in \mathbb{R}$ such that

$$(q, r)^\intercal (w, w^*) > 0 \text{ and } (q, r)^\intercal (a_i, b_i) \leq 0,\ \forall 1 \leq i \leq m.$$

Equivalently,

$$q^\intercal w > -rw^* \text{ and } q^\intercal a_i \leq -rb_i,\ \forall 1 \leq i \leq m.$$

Which is same as

$$w^\intercal q > -rw^* \text{ and } a_i^\intercal q \leq -rb_i,\ \forall 1 \leq i \leq m. \quad (1)$$

Now, we consider two cases depending on the sign of r. In both the cases, we will show that there is a primal feasible solution α with value greater than w^*, contradicting the optimality of α^*.

Case 1: $r < 0$. Divide (1) by $-r$ and define $\alpha = q/-r$. We get

$$a_i^\intercal \alpha \le b_i \ \forall i, \text{ and } w^\intercal \alpha > w^*,$$

which is a contradiction.

Case 2: $r \ge 0$, which means $r + 1 > 0$. Now, consider $\alpha = (r+1)\alpha^* + q$. Since α^* is primal optimal, we know

$$(r+1)a_i^\intercal \alpha^* \le (r+1)b_i, \ \forall i, \text{ and } (r+1)w^\intercal \alpha^* = (r+1)w^*.$$

Combining this with (1), we get

$$a_i^\intercal \alpha = (r+1)a_i^\intercal \alpha^* + a_i^\intercal q \le (r+1)b_i - rb_i = b_i, \ \forall i.$$

And

$$w^\intercal \alpha = (r+1)w^\intercal \alpha^* + w^\intercal q > (r+1)w^* - rw^* = w^*.$$

This gives us a contradiction.

□

3 Various Forms of LP

Primal	Dual
$\max\ w^\intercal x : Ax \le b$	$\min\ b^\intercal y : A^\intercal y = w,\ y \ge 0$
$\max\ w^\intercal x : Ax = b,\ x \ge 0$	$\min\ b^\intercal y : A^\intercal y \ge w$
$\max\ w^\intercal x : Ax \le b,\ x \ge 0$	$\min\ b^\intercal y : A^\intercal y \ge w,\ y \ge 0$